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ASYMPTOTIC REPRESENTATION OF
SOLUTIONS OF THE BASIC SEMI-
CONDUCTOR DEVICE EQUATIONS

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THE BASIC SEMICONDUCTOR DEVICE EQUATIONS

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ABSTRACT

In this paper the basic semiconductor device equations modelling a
symmetric one-dimensional voltage-controlled diode are formulated as a
singularly perturbed two point boundary value problem. The perturbation
parameter is the normed Debye-length of the device. The authors derive the zeroth and
first order terms of the matched asymptotic expansion of the solutions, which
are the sums of uniformly smooth outer terms (reduced solutions) and the
exponentially varying inner terms (layer solutions). The main result of the
paper is that, if the perturbation parameter is sufficiently small then there
exists a solution of the semiconductor device problem which is approximated
uniformly by the zeroth order term of the expansion, even for large applied
voltages. This result shows the validity of the asymptotic expansions of the
solutions of the semiconductor device problem in physically relevant high-
injection conditions.

AMS (MOS) Subject Classifications: 34B15, 34D15, 34E05, 78A35

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SIGNIFICANCE AND EXPLANATION

It is well-known that potential distribution and current flow in a pn-junction diode are described by the solution of a system of ordinary differential equations subject to certain boundary conditions. We scale the system appropriately and obtain a singular perturbation problem, i.e. certain derivatives of the dependent variables are multiplied by a small parameter which is identified as the normed Debye-length of the device.

The singular perturbation character of the problem introduces two different scales of variation of solutions, namely a fast one on which the solutions vary close to the pn-junction and a slow one away from the junction.

We derive separate representations of the solutions which hold inside and outside the pn-layer respectively, and we obtain asymptotic expansions of solutions by matching these local representations.

The main result of this paper is that the asymptotic expansion 'represents' a solution of the semiconductor problem for small λ, i.e. there is a solution which is approximated well by the derived (finite) asymptotic expansions, provided the singular perturbation parameter is small. We present an estimate for the approximation error depending on the applied voltage and show that the singular perturbation approach 'covers' physically relevant bias ranges for modern, highly doped devices.

ASYMPTOTIC REPRESENTATION OF SOLUTIONS OF
THE BASIC SEMICONDUCTOR DEVICE EQUATIONS

Peter A. Markowich* and Christian Schmeiser*

1. INTRODUCTION

This paper is concerned with the asymptotic representation of solutions of the basic semiconductor device equations for the case of a simple model device, namely the symmetric one-dimensional diode.

The physical situation we encounter is as follows. A semiconductor (e.g. silicon) is doped with acceptor ions in the left side (p-side) and with donor ions in the right side (n-side) and a bias $V = V_A - V_C$ is applied to the Ohmic contacts (see Figure 1).
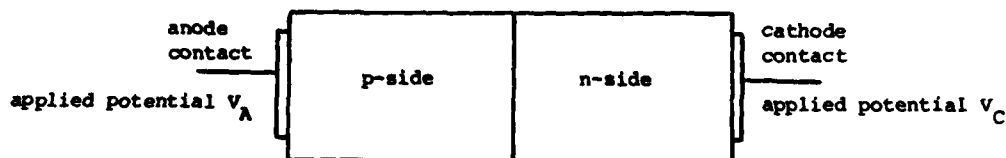
Figure 1: Diode

For simplicity we make the following symmetry assumptions:

(i) The pn-junction (that is the boundary between the n and the p-region) is in the middle of the device.

(ii) The concentration of acceptor atoms in the p-side and the concentration of donor atoms in the n-side are constant and equal (to $C > 0$, C is called doping concentration).

(iii) The applied potentials V_C, V_A satisfy: $V_C = -V_A$.

Under these assumptions, the performance of the device is described by the following two-point boundary value problem

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(1.1a) $\lambda^2 \psi'' = n - p - 1$

(1.1b) $n' = n\psi' + \dfrac{J}{2}$

$\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} \quad 0 < x < 1$

(1.1c) $p' = -p\psi' - \dfrac{J}{2}$

(1.1d) $J' = 0$

subject to

(1.1e) $\psi(0) = 0, \quad \psi(1) = \psi_{BI}(\delta) - \dfrac{U}{2}, \quad \psi_{BI}(\delta) = \ln\left(\dfrac{1 + \sqrt{1 + 4\delta^4}}{2\delta^2}\right)$

(1.1f) $n(0) = p(0), \quad n(1) = \dfrac{1}{2}\left(1 + \sqrt{1 + 4\delta^4}\right)$

(1.1g) $p(1) = \dfrac{1}{2}\left(-1 + \sqrt{1 + 4\delta^4}\right)$

where the dependent variables have the following physical meaning

ψ: (scaled) electrostatic potential

n: (scaled) electron concentration

p: (scaled) hole concentration

J: (scaled) total current density.

The x-interval $[0,1]$ on which the problem is posed represents the n-side of the device (after scaling), $U = \dfrac{V}{U_T}$ (where $U_T = \dfrac{1}{40}$ Volt is the thermal voltage) represents the voltage parameter, ψ_{BI} is that potential at $x = 1$ which prevails if zero external bias is applied (built-in-potential, originating from the doping) and $\lambda, \delta > 0$.

The problem (1.1) is derived by adapting the basic semiconductor device equations (as given by Van Roosbroeck (1950)) to our specific device using the symmetry assumptions (i), (ii), (iii) and other simplifying assumptions (like constant electron and hole mobilities and neglect of generation-recombination of carriers) and by appropriate scaling (the length of the device is scaled to 2, the doping concentration to 1). Details on the assumptions and on the scaling are given in Markowich and Ringhofer (1984) and Markowich (1983), (1984).

The parameters λ and δ are introduced by the scaling. Physically λ is the normed Debye length of the device, that means

$$(1.2a) \qquad \lambda = \sqrt{\frac{\varepsilon U_T}{q \ell^2 C}}$$

where ε is the material permittivity constant, q the elementary charge and 2ℓ the (original) length of the device.

δ^2 is the scaled intrinsic number of the device

$$(1.2b) \qquad \delta^2 = \frac{n_i}{C}$$

(n_i is the number of free electrons of the semiconductor per unit volume, $n_i \approx 10^{10}/cm^3$ for silicon).

ε, U_T, q and n_i are material constants while the length ℓ and the doping concentration C specify the device.

For a realistic silicon diode, we have $C \gtrsim 10^{17}/cm^3$ and $\ell \approx 10^{-2}cm$. This yields $\lambda^2 \lesssim 10^{-7}$. Thus the problem (1.1) can be regarded as singularly perturbed two-point boundary value problem with perturbation parameter λ and the standard method of matched asymptotic expansion can be employed. It turns out that a boundary layer (i.e. a small region of fast variation) occurs in ψ, n, p at the junction $x = 0$. No layer occurs at the Ohmic contact $x = 1$, those solutions which are approximated by the asymptotic expansion are uniformly (in λ) smooth away from $x = 0$.

Also δ is small (practically $\delta^2 \lesssim 10^{-7}$). We will, however, see later on that the smallness of δ has a weaker impact on the solution structure than the smallness of λ (δ^2 is not a singular perturbation parameter since it does not multiply a derivative of a dependent variable).

In the recent past, many papers dealt with the singular perturbation analysis of the basic semiconductor device equations (e.g. Vasileva and Butuzov (1978), Vasileva and Stelmakh (1977), Markowich and Ringhofer (1984), Markowich (1984)). The authors of these papers concentrate on deriving the zeroth order terms of the asymptotic expansions of the

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solutions of the semiconductor device equations as $\lambda \to 0+$ (even in the high dimensional case and for devices much more complicated than our symmetric diode). However, the question of the validity of these asymptotic expansions is a rather unsettled issue. It was proven by Markowich (1984) that for zero applied bias $(U = 0)$ there is a solution of the semiconductor device problem which is close to the zeroth order term of the expansion if λ is small (even in the multidimensional case). For the one-dimensional case this result was carried over to small applied bias $(|U| < c$ where c tends to zero rapidly as $\lambda \to 0+$, $\delta \to 0+$; see Markowich, Ringhofer, Selberherr and Langer (1982)). However, this is of extremely limited practical applicability, since it only means that for sufficiently high doping or for a sufficiently large device (large C or large ℓ implies small λ) and for biasing conditions sufficiently close to thermal equilibrium (represented by $U = 0$) the solutions are asymptotically represented by the asymptotic expansions. In practice, however, one is interested in the performance of highly doped devices when high voltages are applied.

In this paper we show - at least for our simple model device - that there is a solution of (1.1) which is approximated by the zeroth order terms of the asymptotic expansion if λ is sufficiently small and if U is in some specified voltage range which gets larger as $\lambda \to 0+$ and which includes physically relevant high-injection biasing conditions. This result in some sense justifies the singular perturbation approach for the semiconductor device equations.

The paper is organized as follows. Section 2 contains the derivation of the zeroth and first order terms of the asymptotic expansions, the main result of the paper and the functional analytic framework employed for its proof are given in Section 3. The highly technical details of the proof are collected in the Appendices A and B.

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2. DERIVATION OF ASYMPTOTIC EXPANSIONS

We will now apply the approach of matched asymptotic expansions to the problem (1.1). Therefore we assume that the solution (ψ, n, p, J) has an asymptotic expansion in powers of λ where each coefficient is the sum of a term which is independent of λ and of boundary layer terms. It was shown by Markowich and Ringhofer (1984) that the terms representing the layer at $x = 1$ (at the Ohmic contact) vanish in zeroth and first order. Thus we expect a layer only to occur at the junction $x = 0$ and make the ansatz:

$$\psi(x, \lambda) = \bar{\psi}(x) + \hat{\psi}(\tau) + \lambda\bar{\psi}_1(x) + \lambda\hat{\psi}_1(\tau) + \ldots$$

$$n(x, \lambda) = \bar{n}(x) + \hat{n}(\tau) + \lambda\bar{n}_1(x) + \lambda\hat{n}_1(\tau) + \ldots$$

(2.1)

$$p(x, \lambda) = \bar{p}(x) + \hat{p}(\tau) + \lambda\bar{p}_1(x) + \lambda\hat{p}_1(\tau) + \ldots$$

$$J(x) = \bar{J} + \lambda\bar{J}_1 + \ldots$$

where the dots denote terms of order at least λ^2. Note that the current density J is independent of x because of (1.1d). $\tau = \frac{x}{\lambda}$ is the fast independent variable. The terms marked with '$-$' are independent of λ and the terms marked with '$\wedge$' are zeroth and first order layer terms which are required to decay to zero as the fast independent variable tends to infinity:

(2.2)
$$\hat{\psi}(\infty) = \hat{\psi}_1(\infty) = \hat{n}(\infty) = \hat{n}_1(\infty) = \hat{p}(\infty) = \hat{p}_1(\infty) = 0 .$$

By inserting (2.1) into (1.1) and by equating coefficients of equal powers of λ we will obtain boundary value problems for the terms in (2.1).

Construction of zeroth order terms:

By comparing coefficients of λ^0 and evaluating away from $x = 0$ we obtain the reduced equations

$$0 = \bar{n} - \bar{p} - 1 ,$$

$$\bar{n}' = \bar{n}\bar{\psi}' + \frac{\bar{J}}{2} ,$$

$$\bar{p}' = -\bar{p}\bar{\psi}' - \frac{\bar{J}}{2} ,$$

or equivalently

$$\bar{n} = \bar{p} + 1 ,$$

$$(2.3) \qquad \bar{\psi}' = - \frac{\bar{J}}{2\bar{p} + 1} ,$$

$$\bar{p}' = - \frac{\bar{J}}{2(2\bar{p} + 1)} .$$

For the zeroth order boundary layer terms we get (by comparing coefficients of λ^0 close to $x = 0$)

$$\ddot{\hat{\psi}} = \hat{n} - \hat{p} ,$$

$$(2.4) \qquad \dot{\hat{n}} = (\bar{n}(0) + \hat{n})\dot{\hat{\psi}} ,$$

$$\dot{\hat{p}} = -(\bar{p}(0) + \hat{p})\dot{\hat{\psi}} ,$$

where the dot denotes differentiation with respect to τ. Equating zeroth order terms in the boundary conditions (1.1e) – (1.1g) gives

$$\bar{\psi}(0) + \hat{\psi}(0) = 0, \quad \bar{n}(0) + \hat{n}(0) = \bar{p}(0) + \hat{p}(0) ,$$

$$(2.5)$$

$$\bar{\psi}(1) = \psi_{BI} - \frac{U}{2}, \quad \bar{n}(1) = p_1 + 1, \quad \bar{p}(1) = p_1 .$$

By conditions (2.2), $\hat{\psi}, \hat{n}, \hat{p}$ have to be trajectories on the stable manifold of (2.4) which is represented by

$$\hat{n} = \bar{n}(0)(e^{\hat{\psi}} - 1) ,$$

$$(2.6) \qquad \hat{p} = \bar{p}(0)(e^{-\hat{\psi}} - 1) ,$$

$$\dot{\hat{\psi}} = - \sqrt{2(\hat{n} + \hat{p} - \hat{\psi})} \; \mathrm{sgn} \; \hat{\psi}(0) ,$$

(see Schmeiser and Weiss (1984)).

Using (2.6) we obtain the reduced boundary conditions from (2.5)

$$(2.7) \qquad \bar{n}(0)e^{-\bar{\psi}(0)} = \bar{p}(0)e^{\bar{\psi}(0)}, \quad \bar{\psi}(1) = \psi_{BI} - \frac{U}{2}, \quad \bar{p}(1) = p_1 .$$

Equations (2.3) can be integrated and the solution of (2.3), (2.7) (reduced solution) can be written as

$$\bar{J} = 2\left(\frac{b}{(b-1)^2} - \delta^4\right) ,$$

$$\bar{p} = \frac{1}{2}\left(-1 + \sqrt{\left(\frac{b+1}{b-1}\right)^2 - 2\bar{J}x}\right) ,$$

(2.8)

$$\bar{n} = \bar{p} + 1 ,$$

$$\bar{\psi} = 2\bar{p} + \ell n \sqrt{b} - \frac{2}{b-1} ,$$

where the parameter $b > 1$ is related to U by

(2.9)
$$\frac{U}{2} - \psi_{BI} + 2p_1 = \frac{2}{b-1} - \ell n \sqrt{b} =: F(b)$$

Obviously

$$\lim_{b\downarrow 1} F(b) = \infty, \quad \lim_{b\to\infty} F(b) = -\infty, \quad F'(b) = -\frac{2}{(b-1)^2} - \frac{1}{2b} < 0, \quad b \in (1,\infty)$$

holds. Thus, (2.9) is a one-to-one relation between $U \in (-\infty,\infty)$ and $b \in (1,\infty)$.

From (2.8) and (2.9) we get a first order (formal) asymptotic approximation of the voltage-current-characteristic:

(2.10) $$\frac{U}{2} = \ell n \frac{1 + \sqrt{1 + 4\delta^4}}{2\delta^2} - \ell n \frac{1 + \sqrt{1 + 4\delta^4 + 2\bar{J}}}{\sqrt{4\delta^4 + 2\bar{J}}} - \sqrt{1 + 4\delta^4} + \sqrt{1 + 4\delta^4 + 2\bar{J}}$$

We compute $\bar{n}(0), \bar{p}(0)$ in terms of b by using (2.8) and rewrite (2.6) as

(2.11a)
$$\hat{n} = \frac{b}{b-1}(e^{\hat{\psi}} - 1) ,$$

(2.11b)
$$\hat{p} = \frac{1}{b-1}(e^{-\hat{\psi}} - 1) ,$$

(2.11c)
$$\dot{\hat{\psi}} = \sqrt{2\left(\frac{b}{b-1}e^{\hat{\psi}} + \frac{1}{b-1}e^{-\hat{\psi}} - \hat{\psi} - \frac{b+1}{b-1}\right)} ,$$

(2.11d)
$$\hat{\psi}(0) = \bar{\psi}(0) = -\ell n \sqrt{b} .$$

(2.11c) is obtained by observing that $\text{sgn}\,\dot{\hat{\psi}}(0) = -1$ (since $\text{sgn}\,\dot{\bar{\psi}}(0) = 1$ and $\dot{\bar{\psi}}(0) + \dot{\hat{\psi}}(0) = 0$). The unique solvability of (2.11c,d) and the exponential decay of $\hat{\psi}, \hat{n}$ and $\hat{p}$ follow from an application of the theory of Fife (1974) to the second order equation in (2.4) (see Markowich, Ringhofer, Selberherr and Langer (1982)).

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Construction of first order coefficients:

Generally n-th order coefficients of matched asymptotic expansions are defined by problems which are linearized versions of the problem defining the zeroth order coefficients with inhomogeneities only depending on coefficients of order at most $(n - 1)$ (see Schmeiser and Weiss (1984)). The first order coefficients satisfy

$$\bar{n}_1 = \bar{p}_1 ,$$

(2.12)
$$\bar{\psi}_1' = - \frac{1}{2\bar{p} + 1} \bar{J}_1 + \frac{2\bar{J}}{(2\bar{p} + 1)^2} \bar{p}_1 ,$$

$$\bar{p}_1' = - \frac{1}{2(2\bar{p} + 1)} \bar{J}_1 + \frac{\bar{J}}{(2\bar{p} + 1)^2} \bar{p}_1 ,$$

for the part which is independent of λ and

$$\ddot{\hat{\psi}}_1 = \hat{n}_1 - \hat{p}_1 ,$$

(2.13)
$$\dot{\hat{n}}_1 = \frac{b}{b - 1} e^{\hat{\psi}} \dot{\hat{\psi}}_1 + \dot{\hat{\psi}}\hat{n}_1 + \dot{\hat{\psi}}\bar{p}_1(0) + \hat{n}\bar{\psi}'(0) + \hat{\psi}\tau\bar{p}'(0) ,$$

$$\dot{\hat{p}}_1 = - \frac{1}{b - 1} e^{-\hat{\psi}} \dot{\hat{\psi}}_1 - \dot{\hat{\psi}}\hat{p}_1 - \dot{\hat{\psi}}\bar{p}_1(0) - \hat{p}\bar{\psi}'(0) - \hat{\psi}\tau\bar{p}'(0) ,$$

for the layer terms. The boundary conditions

$$\bar{\psi}_1(0) + \hat{\psi}_1(0) = 0, \quad \bar{n}_1(0) + \hat{n}_1(0) = \bar{p}_1(0) + \hat{p}_1(0)$$

(2.14)
$$\bar{\psi}_1(1) = 0, \quad \bar{n}_1(1) = 0, \quad \bar{p}_1(1) = 0 .$$

hold. The results of Appendix A(a) imply that decaying solutions of (2.13) can be written in the form

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$$\hat{\psi}_1 = c\,\frac{\dot{\hat{\psi}}}{\dot{\hat{\psi}}(0)} + \hat{\psi}_p \,,$$

(2.15)
$$\hat{n}_1 = \frac{b}{b-1}\, e^{\hat{\psi}}\hat{\psi}_1 + \bar{p}_1(0)(e^{\hat{\psi}} - 1) + \hat{n}_p \,,$$

$$\hat{p}_1 = -\frac{1}{b-1}\, e^{-\hat{\psi}}\hat{\psi}_1 + \bar{p}_1(0)(e^{-\hat{\psi}} - 1) + \hat{p}_p \,,$$

where c is a constant and

$$\hat{n}_p = -\bar{p}'(0)\int_\tau^\infty e^{\hat{\psi}(\tau)-\hat{\psi}(s)}(2\hat{n}(s) + s\dot{\hat{\psi}}(s))ds$$

(2.16)
$$\hat{p}_p = -\bar{p}'(0)\int_\tau^\infty e^{-\hat{\psi}(\tau)+\hat{\psi}(s)}(2\hat{p}(s) + s\dot{\hat{\psi}}(s))ds$$

$$\hat{\psi}_p = -\dot{\hat{\psi}}(\tau)\int_0^\tau \dot{\hat{\psi}}^{-2}(s)\int_s^\infty \dot{\hat{\psi}}(u)\left[\bar{p}_1(0)(e^{\hat{\psi}(u)} - e^{-\hat{\psi}(u)}) + \hat{n}_p(u) - \hat{p}_p(u)\right]du\,ds$$

Integration of (2.12) using (2.14) and (2.15) gives expressions for the reduced solution of order 1

$$\bar{J}_1 = \frac{2\sqrt{b}}{b+1}\,(\hat{n}_p(0) - \hat{p}_p(0)) \,,$$

(2.17)
$$\bar{n}_1 = \bar{p}_1 = \bar{J}_1\,\frac{1-x}{2(2\bar{p}+1)} \,, \quad \bar{\psi}_1 = 2\bar{p}_1 \,.$$

The constant in (2.15) is $c = \dfrac{2\sqrt{b}\,(b-1)}{(b+1)^2}\,(\hat{p}_p(0) - \hat{n}_p(0))$. The expressions (2.15), (2.17) can be simplified using the following

Lemma 2.1: $\hat{n}_p(0) - \hat{p}_p(0) = -\dfrac{\bar{J}(b-1)}{2\sqrt{b}}\,\dot{\hat{\psi}}(0)$

Proof: Relations (2.3) and (2.8) imply

$$\bar{p}'(0) = -\frac{\bar{J}(b-1)}{2(b+1)} \,.$$

Thus, we get

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$$\hat{n}_p(0) = \frac{\bar{J}(b-1)}{2\sqrt{b}\,(b+1)} \int_0^\infty e^{-\hat{\psi}(s)}(2\hat{n}(s) + s\dot{\hat{\psi}}(s))ds =$$

$$= \frac{\bar{J}(b-1)}{2\sqrt{b}\,(b+1)} \left[\int_0^\infty e^{-\hat{\psi}(s)}\,\frac{2b}{b-1}\,(e^{\hat{\psi}(s)} - 1)ds + \lim_{\tau\to\infty} \int_0^\tau s e^{-\hat{\psi}(s)}\dot{\hat{\psi}}(s)ds \right]$$

By partial integration in the second integral we obtain

$$\hat{n}_p(0) = \frac{\bar{J}(b-1)}{2\sqrt{b}\,(b+1)} \left[-2b \int_0^\infty \hat{p}(s)ds + \lim_{\tau\to\infty} \left(-\tau e^{-\hat{\psi}(\tau)} + \int_0^\tau e^{-\hat{\psi}(s)}ds \right) \right]$$

$$= \frac{\bar{J}(b-1)}{2\sqrt{b}\,(b+1)} \left[-2b \int_0^\infty \hat{p}(s)ds + \lim_{\tau\to\infty} \left(-\tau(e^{-\hat{\psi}(\tau)} - 1) + \int_0^\tau (e^{-\hat{\psi}(s)} - 1)ds \right) \right]$$

$$= \frac{\bar{J}(b-1)}{2\sqrt{b}\,(b+1)} \left[-2b \int_0^\infty \hat{p}(s)ds + (b-1) \int_0^\infty \hat{p}(s)ds \right] = -\frac{\bar{J}(b-1)}{2\sqrt{b}} \int_0^\infty \hat{p}(s)ds .$$

Similarly we show

$$\hat{p}_p(0) = -\frac{\bar{J}(b-1)}{2\sqrt{b}} \int_0^\infty \hat{n}(s)ds .$$

Hence, from (2.4) we obtain

$$\hat{n}_p(0) - \hat{p}_p(0) = \frac{\bar{J}(b-1)}{2\sqrt{b}} \int_0^\infty (\hat{n}(s) - \hat{p}(s))ds = -\frac{\bar{J}(b-1)}{2\sqrt{b}} \dot{\hat{\psi}}(0) .$$

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3. THE REPRESENTATION THEOREM

The question to be answered now is whether and how well do the zeroth order terms of the asymptotic expansions (2.1) approximate a solution (ψ, n, p, J) of the problem (1.1).

The following theorem is derived by a straight forward application of the representation theory for singularly perturbed two-point-boundary value problems given in Schmeiser and Weiss (1984).

Theorem 3.1: For every $U \in R$ and $\delta > 0$ there is $\lambda_0 = \lambda_0(U,\delta) > 0$ such that for $\lambda < \lambda_0(U,\delta)$ there is a locally unique solution (ψ, n, p, J) of (1.1) which satisfies

$$\psi(x,\lambda) = \bar{\psi}(x) + \hat{\psi}\left(\frac{x}{\lambda}\right) + 0(\lambda)$$

$$n(x,\lambda) = \bar{n}(x) + \hat{n}\left(\frac{x}{\lambda}\right) + 0(\lambda)$$

$$p(x,\lambda) = \bar{p}(x) + \hat{p}\left(\frac{x}{\lambda}\right) + 0(\lambda)$$

$$J = \bar{J} + 0(\lambda)$$

uniformly for $x \in [0,1]$.

Also the existence of asymptotic approximations of arbitrary order of accuracy can be concluded from Schmeiser and Weiss (1984).

The theorem says that, for given doping concentration C (which determines δ by (1.2b)) and for a given voltage U there is $\ell_0 > 0$ such that for all diodes with length $2\ell > 2\ell_0$ the corresponding semiconductor device equations (1.1) have solutions which are uniformly approximated (to order λ) by the zeroth order terms of the asymptotic expansion (1.1) (note that ℓ and C determine λ by (1.2a) for a given material and that $\lambda \to 0$ for $\ell \to \infty$ when C is constant).

Practically, this is a rather meaningless result since one is not interested in applying a fixed voltage to various (sufficiently large) devices, but in varying the voltage applied to a fixed highly doped device (that means λ and δ are small and fixed and U varies over a certain voltage range).

To prove an approximation result which is uniform in a sufficiently large U interval we basically proceed as Schmeiser and Weiss (1984) did for general singularly perturbed two-point boundary value problems but we always keep track on how small λ has to be (in dependence of δ and the parameter b which is related to U by (2.9)) in order to guarantee the validity of the results in Schmeiser and Weiss (1984).

We regard the problem (1.1) (after eliminating (1.1d), putting all terms on the left hand side and expressing U as function of b, δ by using (2.9)) as operator equation

$$(3.1a) \quad F_{\lambda,\delta,b}(\psi,n,p,J) = (\lambda^2\psi'' - (n-p-1), n'- n\psi' - \frac{J}{2}, \; p'+ p\psi'+\frac{J}{2}, \; \psi(0),\psi(1) - \psi_{BI} + \frac{U(b)}{2},$$

$$n(0) - p(0), n(1) - \frac{1}{2}\left(1 + \sqrt{1 + 4\delta^4}\right), p(1) - \frac{1}{2}\left(-1 + \sqrt{1 + 4\delta^4}\right)) = 0$$

$$(3.1b) \quad F_{\lambda,\delta,b} : B_{1,\lambda} \to B_{2,\lambda} .$$

The following spaces will be needed in the sequel:

$$(3.2a) \qquad\qquad B_{1,\lambda} = W^{2,\infty}(0,1) \times (C[0,1] \cap W^{1,1}(0,1))^2 \times R$$

$$(3.2b) \qquad\qquad B_{2,\lambda} = L^{\infty}(0,1) \times (L^1(0,1))^2 \times R^5$$

equipped with the norms $\|\cdot\|_{1,\lambda}$, $\|\cdot\|_{2,\lambda}$ resp. defined by

$$(3.3a) \quad \|(\psi,n,p,J)\|_{1,\lambda} := \|\psi\|_{L^{\infty}(0,1)} + \lambda\|\psi'\|_{L^{\infty}(0,1)} + \lambda^2\|\psi''\|_{L^{\infty}(0,1)}$$

$$+ \|n\|_{L^{\infty}(0,1)} + \lambda\|n'\|_{L^1(0,1)} + \|p\|_{L^{\infty}(0,1)} + \lambda\|p'\|_{L^1(0,1)} + |J|$$

for $(\psi,n,p,J) \in B_{1,\lambda}$ and

$$(3.3b) \qquad \|(u,v,w,\alpha)\|_{2,\lambda} := \|u\|_{L^{\infty}(0,1)} + \|v\|_{L^1(0,1)} + \|w\|_{L^1(0,1)} + |\alpha|_{\infty}$$

for $(u,v,w,\alpha) \in B_{2,\lambda}$.

$W^{m,p}(0,1)$ denotes the Sobolev space of real valued functions defined on $(0,1)$ whose weak derivatives of order up to m are p-integrable for $1 \le p < \infty$ and essentially bounded for $p = \infty$ (see Adams (1975)); i.e. $f \in W^{m,p}(0,1)$ iff $f^{(i)} \in L^p(0,1)$ for $0 \le i \le m$. We denoted

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$$(3.4) \quad \|f\|_{L^p(0,1)} := \left(\int_0^1 |f(x)|^p dx\right)^{1/p}, \quad 1 \leq p < \infty, \quad \|f\|_{L^\infty(0,1)} := \sup_{x \in [0,1]} |f(x)| .$$

and $|a|_\infty := \max_{1 \leq i \leq n} |a_i|$ for $a = (a_1,\ldots,a_n) \in \mathbb{R}^n$. Also the induced matrix norm is denoted by $|\cdot|_\infty$.

From Sobolev's imbedding theorem (see Adams (1975)) we conclude that $f \in W^{1,1}(0,1)$ implies $f \in C_B(0,1)$ (f is continuous and bounded in $(0,1)$) and that $\|f\|_{L^\infty(0,1)} \leq$ const $\left(\|f\|_{L^1(0,1)} + \|f'\|_{L^1(0,1)}\right)$ holds for all $f \in W^{1,1}(0,1)$. Therefore the functional $\|f\|_{L^\infty(0,1)} + \lambda \|f'\|_{L^1(0,1)}$ defines a norm (for $\lambda > 0$) on $C[0,1] \cap W^{1,1}(0,1)$ under which this space is a Banach space.

We denote the asymptotic expansion (2.1) (up to the first order term) by

$$(3.5a) \qquad \psi_a(x,\lambda) = \bar{\psi}(x) + \hat{\psi}\left(\frac{x}{\lambda}\right) + \lambda \bar{\psi}_1(x) + \lambda \hat{\psi}_1\left(\frac{x}{\lambda}\right)$$

$$(3.5b) \qquad n_a(x,\lambda) = \bar{n}(x) + \hat{n}\left(\frac{x}{\lambda}\right) + \lambda \bar{n}_1(x) + \lambda \hat{n}_1\left(\frac{x}{\lambda}\right)$$

$$(3.5c) \qquad p_a(x,\lambda) = \bar{p}(x) + \hat{p}\left(\frac{x}{\lambda}\right) + \lambda \bar{p}_1(x) + \lambda \hat{p}_1\left(\frac{x}{\lambda}\right)$$

$$(3.5d) \qquad J_a(\lambda) = \bar{J} + \lambda \bar{J}_1 .$$

The terms on the right hand side of (3.5) were computed (up to $\hat{\psi}$ which solves (2.11c,d)) in Section 2. Clearly ψ_a, n_a, p_a and J_a depend on δ and on the parameter b.

For the representation proof we proceed as follows: (A) We compute the residual at (ψ_a, n_a, p_a, J_a)

$$(3.6) \qquad\qquad r(\lambda,\delta,b) = F_{\lambda,\delta,b}(\psi_a, n_a, p_a, J_a)$$

and estimate $\|r(\lambda,\delta,b)\|_{2,\lambda} \leq R(\lambda,\delta,b)$.

(B) We compute the Frechet derivative of $F_{\lambda,\delta,b}$ at (ψ_a, n_a, p_a, J_a):

$$(3.7) \qquad D_{(\psi,n,p,J)} F_{\lambda,\delta,b}(\psi_a, n_a, p_a, J_a) : B_{1,\lambda} \to B_{2,\lambda}$$

and estimate its inverse $\|(D_{(\psi,n,p,J)} F_{\lambda,\delta,b}(\psi_a, n_a, p_a, J_a))^{-1}\|_{B_{2,\lambda} \to B_{1,\lambda}} \leq K_1(\lambda,\delta,b)$.

(C) We estimate the Lipschitz constant of $D_{\psi,n,p,J}F_{\lambda,\delta,b} : B_{1,\lambda} \to L(B_{1,\lambda},B_{2,\lambda})$ in a sufficiently large sphere about (ψ_a,n_a,p_a,J_a):

(3.8)
$$\|D_{\psi,n,p,J}F_{\lambda,\delta,b}(\psi,n,p,J) - D_{\psi,n,p,J}F_{\lambda,\delta,b}(\psi_a,n_a,p_a,J_a)\|_{B_{1,\lambda}\to B_{2,\lambda}}$$

$$\leq K_2(\lambda,\delta,b)\|(\psi - \psi_a, n - n_a, p - p_a, J - J_a)\|_{1,\lambda}$$

for $\|(\psi - \psi_a, n - n_a, p - p_a, J - J_a)\|_{1,\lambda} \leq \rho$.

The bounds R, K_1 and K_2 are explicitly determined in the following

Lemma 3.1. There are constants $\lambda_0, \delta_0, C_1, C_2, C_3, \sigma > 0$ such that the estimates

(3.9)
$$R(\lambda,\delta,b) \leq C_1 \lambda^2 \left[\frac{1}{(b-1)^4} + (\ln \tfrac{1}{\lambda})\delta^8 (\ln \sqrt{b})^{11/2}\right]$$

(3.10)
$$K_2(\lambda,\delta,b) \leq \frac{C_3}{\lambda}$$

hold independently of the radius ρ for $0 < \lambda < \lambda_0$, $0 < \delta < \delta_0$. If, additionally

(3.11)
$$\lambda\left(\ln \tfrac{1}{\lambda}\right)\left[\frac{1}{(b-1)^{16}} + \delta^8 b(\ln \sqrt{b})^{19/2}\right] \leq \sigma$$

then

(3.12)
$$K_1(\lambda,\delta,b) \leq C_2\left[\frac{1}{(b-1)^6} + (\ln \sqrt{b})^2\right]$$

holds, too.

Proof: The estimate (3.9) is proven in Appendix B, (B.1); (3.12) in Appendix A, Lemma A.4 (which uses (3.11)) and (3.10) follows immediately by linearizing $F_{\lambda,\delta,b}$ at (ψ_a,n_a,p_a,J_a).

We will employ the following version of the implicit function theorem (see Chow and Hale (1982)):

Lemma 3.2. Let F be a Lipschitz-continuously Frechet differentiable map from the Banach space B_1 into the Banach space B_2. Assume that the equation $F(x) = r$ has a solution x_a which is isolated, i.e. the Frechet derivative $D_x F(x_a) : B_1 \to B_2$ is boundedly invertible. Let the constants K_1 and K_2 satisfy

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$$\|(D_x F(x_a))^{-1}\|_{B_2 \to B_1} \le K_1 ,$$

$$\|D_x F(x) - D_x F(x_a)\|_{B_1 \to B_2} \le K_2 \|x - x_a\|_{B_1} \quad \text{if} \quad \|x - x_a\| \le \rho \quad \text{for some} \quad \rho > 0 .$$

Then the problem $F(x) = 0$ has a solution x which satisfies

(3.14) $$\|x_a - x\|_{B_1} \le 2K_1 \|r\|_{B_2}$$

if

(3.15) $$\|r\|_{B_2} \le \min\left(\frac{\rho}{2K_1}, \frac{1}{2K_1^2 K_2}\right)$$

holds. The solution x is unique in the sphere with radius $\frac{1}{2K_1 K_2}$ and center x_a.

The main result of the paper is the following:

Theorem 3.1. There are constants $\lambda_1, \delta_1, C_+, C_-, D_+, D_- > 0$ such that the problem (1.1) has a locally unique solution (ψ, n, p, J) for $\lambda \le \lambda_1$, $\delta \le \delta_1$ if either

(i) $\frac{U}{2} > \psi_{BI}$ and $\lambda\left(\frac{U}{2} - \psi_{BI} + 1\right)^{16} \le C_+$

or

(ii) $\frac{U}{2} \le \psi_{BI}$ and $\lambda\left(\ell n \frac{1}{\lambda}\right)\delta^4\left(\psi_{Bi} - \frac{U}{2} + 1\right)^{19/2} e^{-U} \le C_-$

holds. The solution (ψ, n, p, J) satisfies the estimate

(3.16) $$\|\psi - (\bar{\psi} + \hat{\psi})\|_{L^\infty(0,1)} + \|n - (\bar{n} + \hat{n})\|_{L^\infty(0,1)} + \|p - (\bar{p} + \hat{p})\|_{L^\infty(0,1)}$$

$$+ |J - \bar{J}| \le \begin{cases} D_+ \lambda\left(\frac{U}{2} - \psi_{BI} + 1\right)^{1/2} & \text{in the case (i)} \\ D_- \lambda \delta^2\left(\psi_{BI} - \frac{U}{2} + 1\right)^3 e^{-U/2} & \text{in the case (ii)} \end{cases} .$$

Proof: A simple calculation which uses (2.9) shows that (i) and (ii) imply (3.11). Therefore the estimates (3.9), (3.10) and (3.12) of Lemma 3.1 hold and the implicit function theorem (Lemma 3.2) can be applied to conclude the locally unique solvability of $F_{\lambda,\delta,b}(\psi, n, p, J) = 0$. It follows that there is a locally unique solution (ψ, n, p, J) if $R(\lambda, \delta, b) \le \frac{1}{2K_1^2(\lambda, \delta, b)K_2(\lambda, \delta, b)}$. This is guaranteed by (3.11), too. The inequality (3.16) is obtained by estimating the first order terms of the expansion (2.1) (see Appendix A,

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Lemmas A.d. 18, A.d. 20) and by observing that the error bound given by (3.14) (which includes the first order terms) is sharper than (3.16).

We now consider a realistic silicon diode with the numerical values of the parameters given in Section 1. The cases a) and b) in the formulation of Theorem 3.1 yield approximate upper and lower bounds for the applied voltage $V = U_T U$. With $C_+ = C_- = 0(1)$ we obtain

$$(3.17) \qquad -0,2 \text{ Volt} \lesssim V \lesssim 0,8 \text{ Volt}$$

The upper bound represents a large forward bias for the considered device. Thus our theory covers a realistically large forward bias range.

In the reverse bias case application of up to -10 Volt is of practical interest. Apparently the reverse bias range covered by the presented theory is much smaller than desired. We conjecture that this is caused by the fact that the space charge region widenes with increasing reverse bias. This limits the validity of an ansatz which strongly uses the "boundary layer behavior" of the solution.

Figure 3.1 shows the reduced potential $\bar{\psi}$, Figure 3.2 the reduced hole concentration $\bar{p}$ and Figure 3.3 the reduced current density $\bar{J}$ as given by (2.8). The surfaces were obtained by parametrizing the curves $\bar{\psi}, \bar{n}$ on the x-interval $[0,1]$ (n-side of the device) with the scaled applied voltage $\frac{U}{2}$ ranging from $\frac{U}{2} = -10$ (-0.5 volt reverse bias) to $\frac{U}{2} = 40$ (2 volts forward bias). The reduced electron concentration $\bar{n}$ is not depicted since it is given by shifting $\bar{p}$ by the value 1, i.e. $\bar{n} = \bar{p} + 1$. The extension of the reduced solutions to the interval $[-1,1]$ is obtained by using the symmetry conditions

$$\bar{n}(x) = \bar{p}(-x), \ \bar{\psi}(x) = -\bar{\psi}(-x) \ .$$

The reduced current-voltage characteristic $\bar{J} = \bar{J}(U)$ exhibits the well-known exponential behaviour (see Sze (1981)).

The layer-equations (2.11) were solved numerically. Figures 3.4, 3.5, 3.6 show the layer-terms $\hat{\psi}(\tau), \hat{n}(\tau)$ and $\hat{p}(\tau)$ resp., again as surfaces parametrized by $\frac{U}{2}$ varying in the range specified above. Figure 3.4 demonstrates the increase of the layer jump and of the width in the potential ψ as the reverse bias increases. Figures 3.5, 3.6 show the
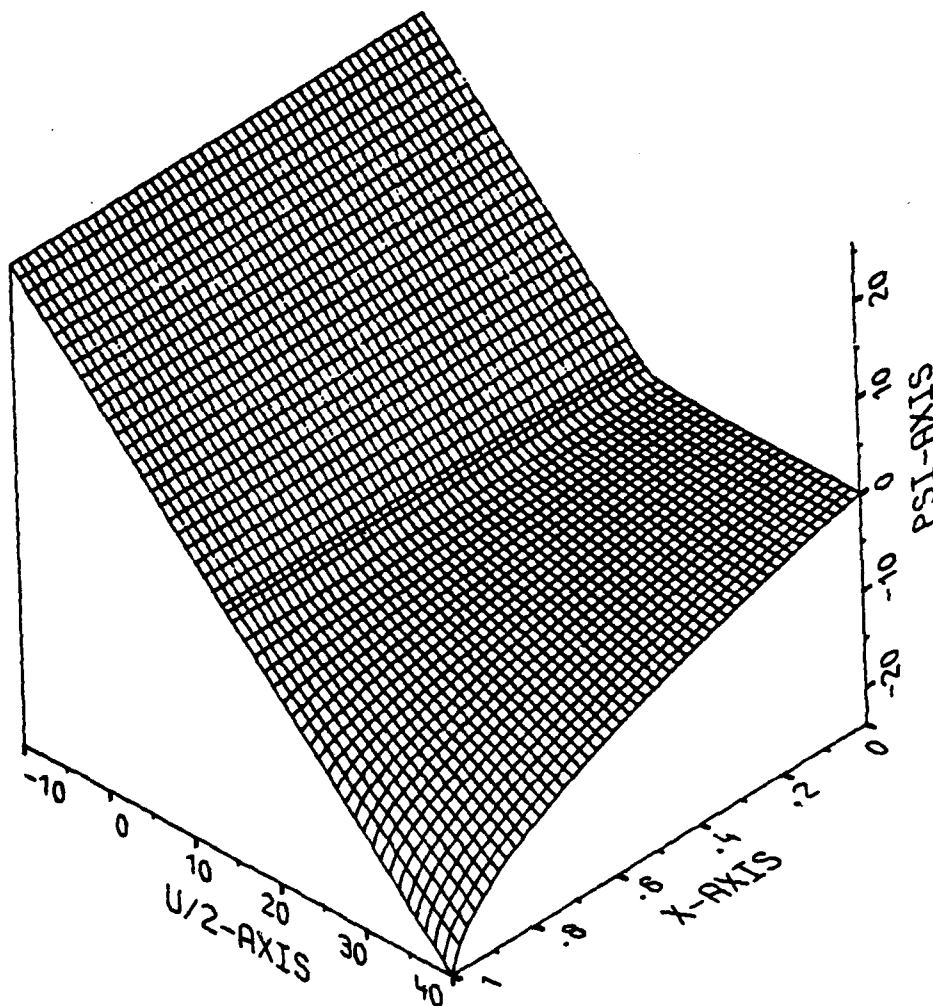
-16-

PSI

Figure 3.1: Reduced Potential $\bar{\psi}(x)$ Parametrized by $\frac{U}{2}$.
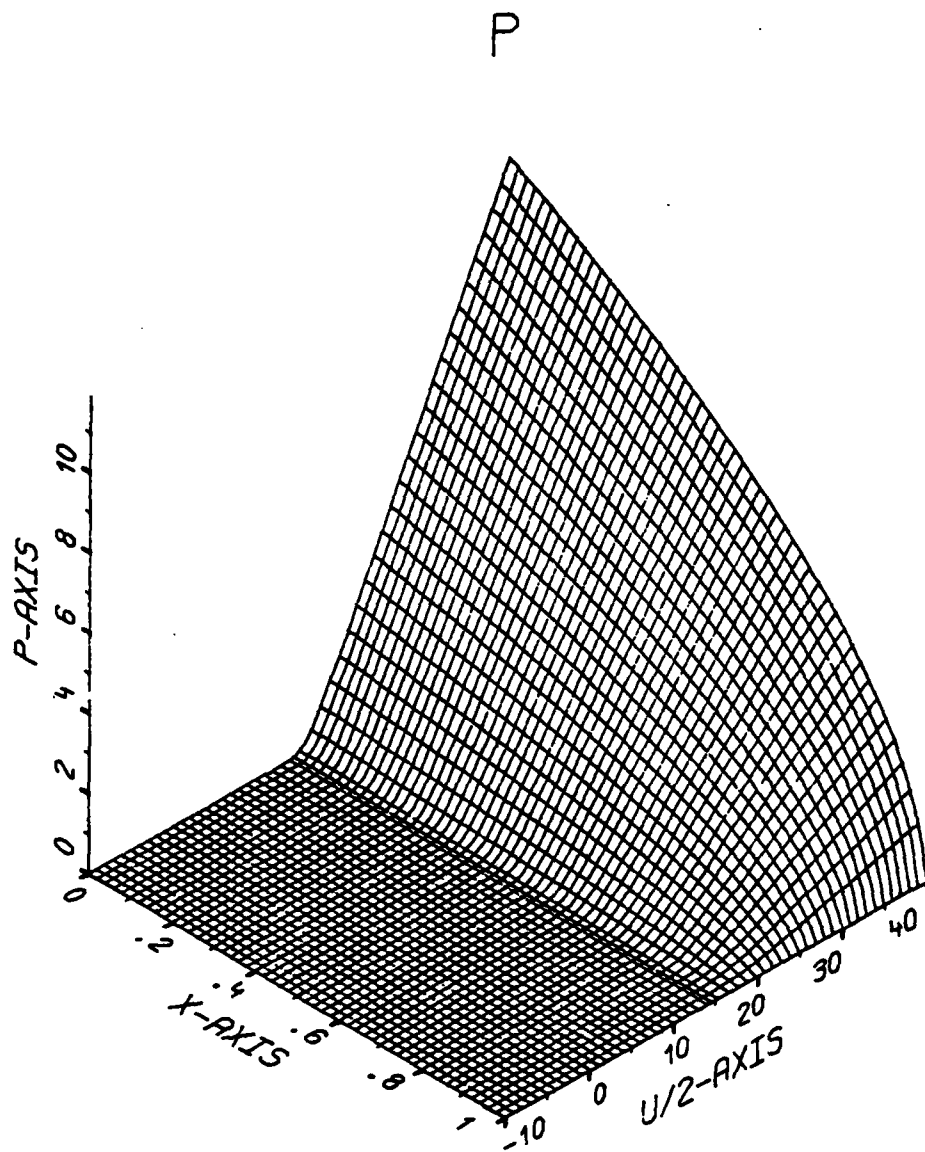
-17-

P

Figure 3.2: Reduced Hole Concentration $\bar{p}(x)$ Parametrized by $\frac{U}{2}$.
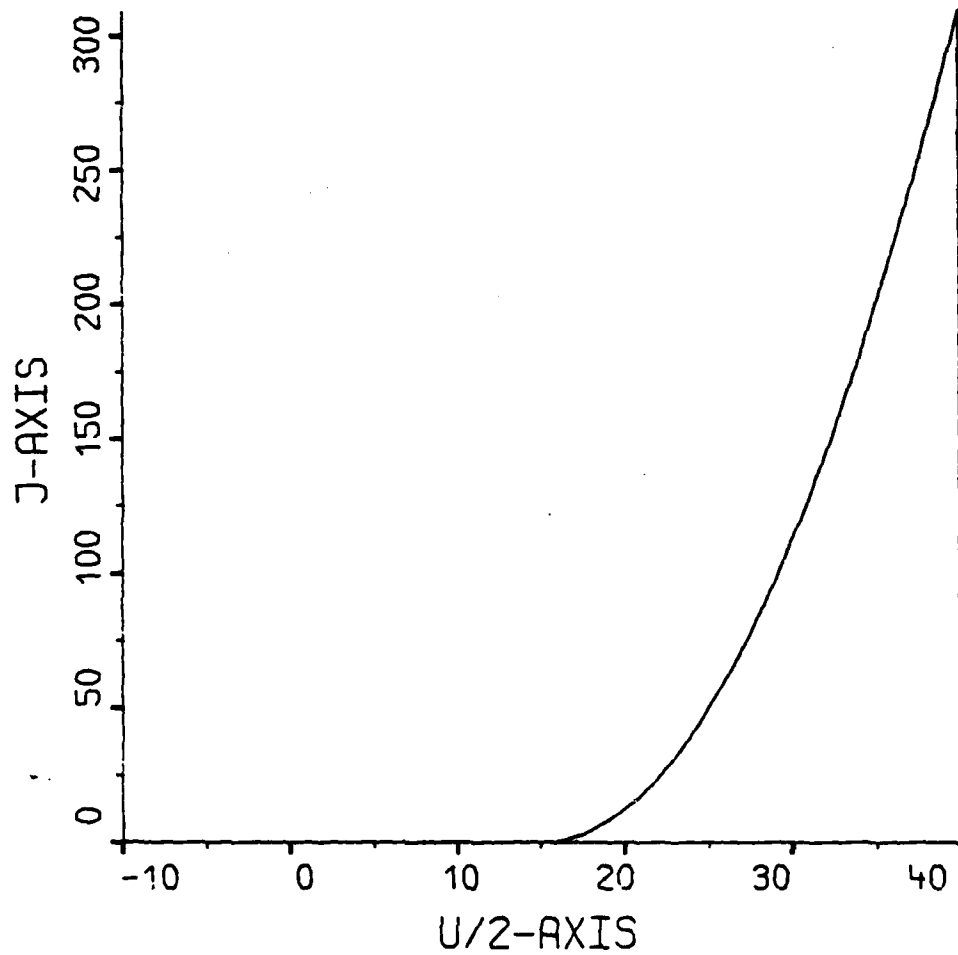
-18-

Figure 3.3: Reduced Current-Voltage Characteristic $\bar{J} = \bar{J}(U)$.
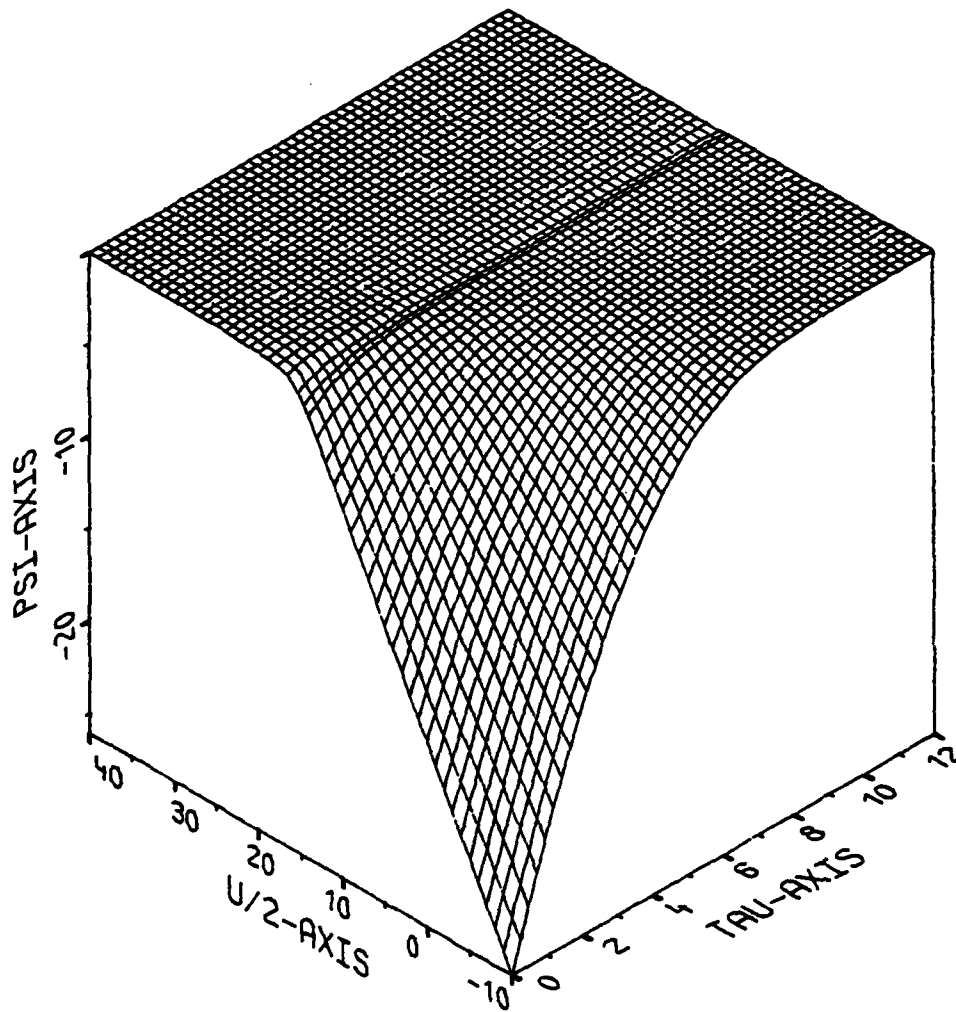
PSI

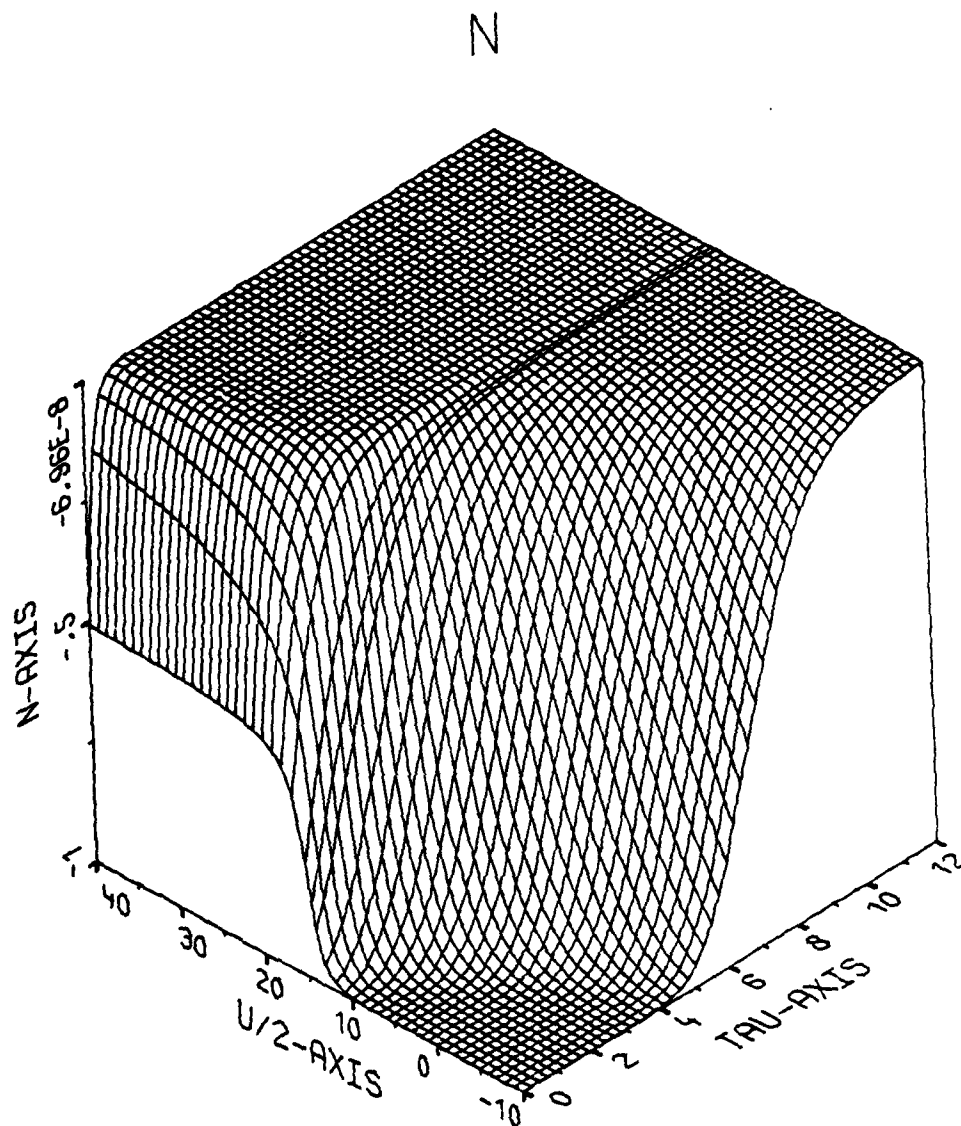Figure 3.4:   Inner Solution $\hat{\psi}(\tau)$ Parametrized by $\frac{U}{2}$.

N

Figure 3.5: Inner Solution $\hat{n}(\tau)$ Parametrized by $\frac{U}{2}$.
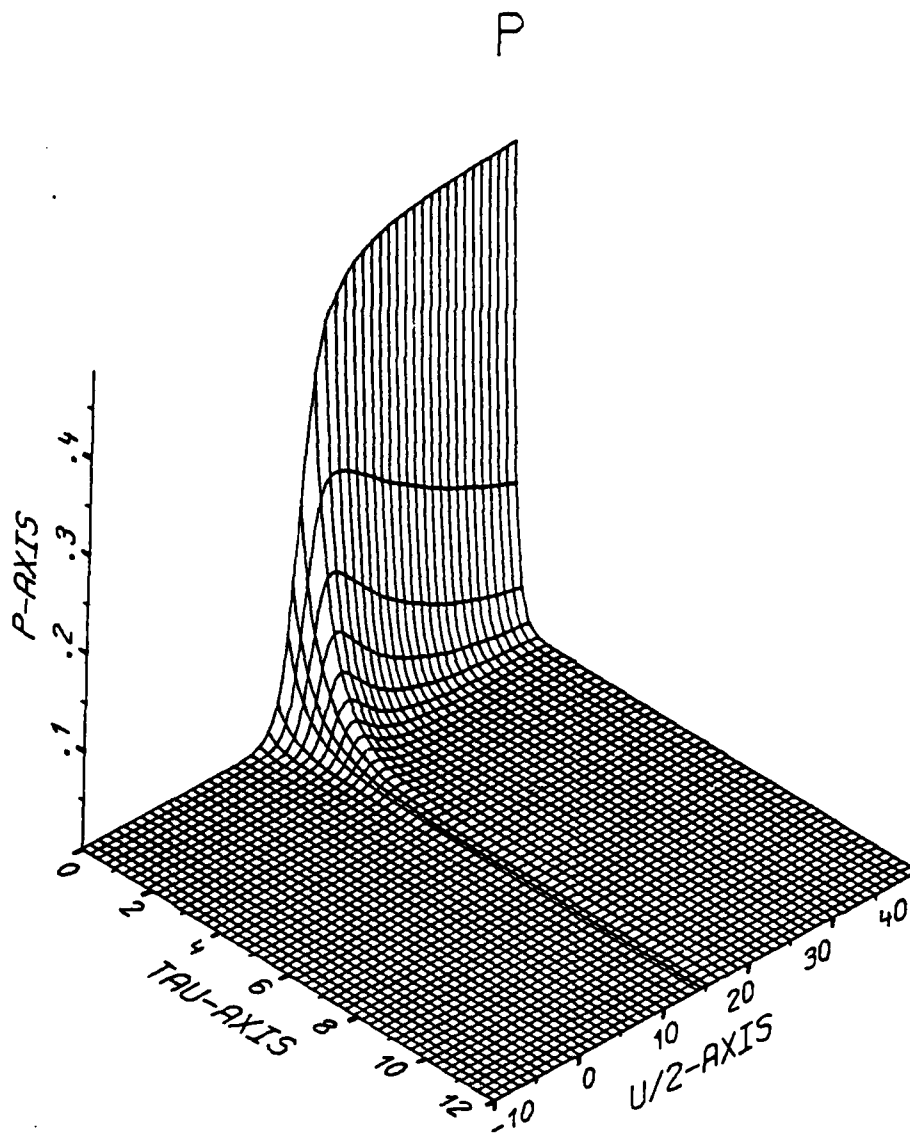
-21-

P

Figure 3.6: Inner Solution $\hat{p}(\tau)$ Parametrized by $\frac{U}{2}$.

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depletion of carriers occuring in the space-charge-region for reverse bias and the increase of the carrier concentration in the space charge region for forward bias.

Also, the 'full' singularly perturbed problem (1.1) was solved numerically (by using the general purpose two-point-boundary value problem COLSYS authored by Ascher, Christiansen and Russell (1978)). Figures 3.7, 3.8, 3.9, 3.10 depict the potential ψ, the electron concentration n, the hole concentration p and the electric field ψ' resp.

A comparison of the full solutions ψ, n, p and the reduced solutions $\bar{\psi}, \bar{n}, \bar{p}$ clearly shows that ψ, n, p are approximated well by $\bar{\psi}, \bar{n}, \bar{p}$ outside the layer-region on the whole considered bias-range, in fact the corresponding reduced and full solutions agree at least up to plot accuracy outside the layer. The current-voltage characteristic $J = J(U)$ is not depicted since it is graphically indistinguishable from the reduced current voltage characteristic $\bar{J} = \bar{J}(U)$ as shown in Figure 3.3. Note that the depicted bias range is larger than (3.17).

Figure 3.10 demonstrates the occurrence of a very large electric field within the layer (note the scale on the PSIP-axis!), which is explained by the asymptotic expansion for ψ':

$$\psi'(x,\lambda) \sim \bar{\psi}'(x) + \frac{1}{\lambda}\overset{\bullet}{\bar{\psi}}\left(\frac{x}{\lambda}\right) + \dots .$$

POTENTIAL

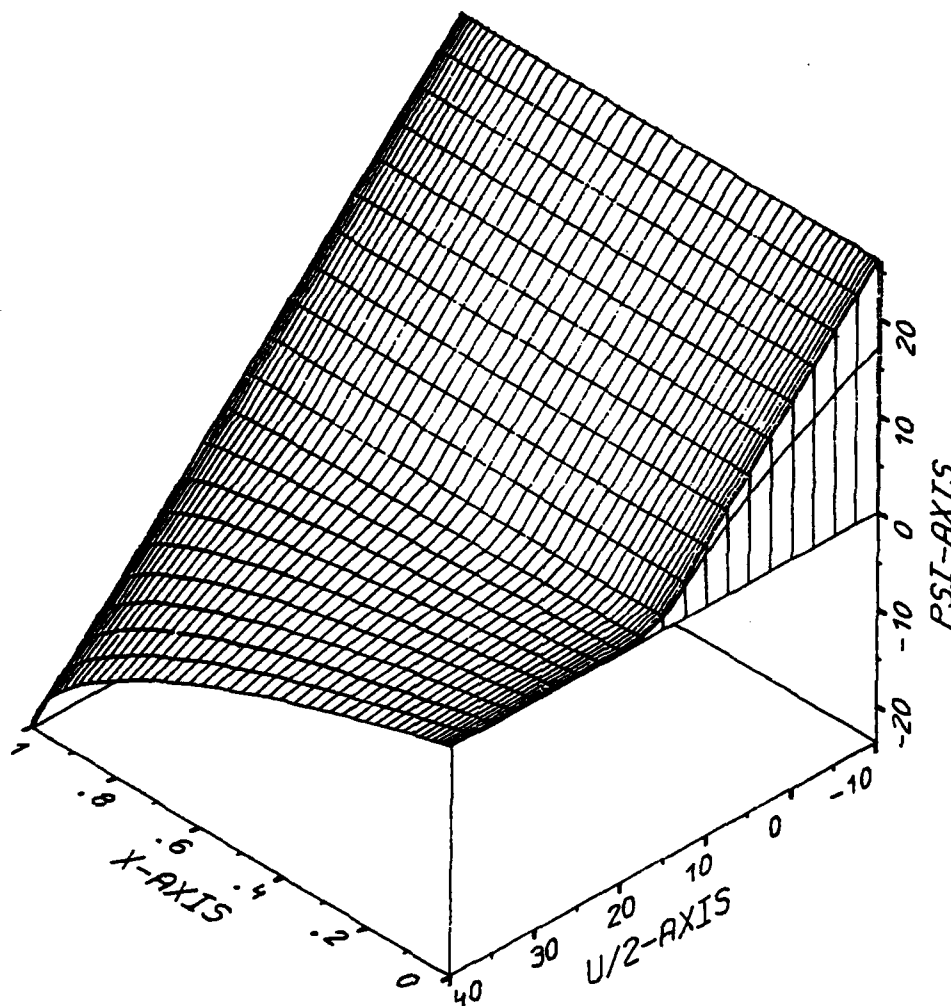Figure 3.7: Potential $\psi(x)$ Parametrized by $\frac{U}{2}$.

ELECTRONS
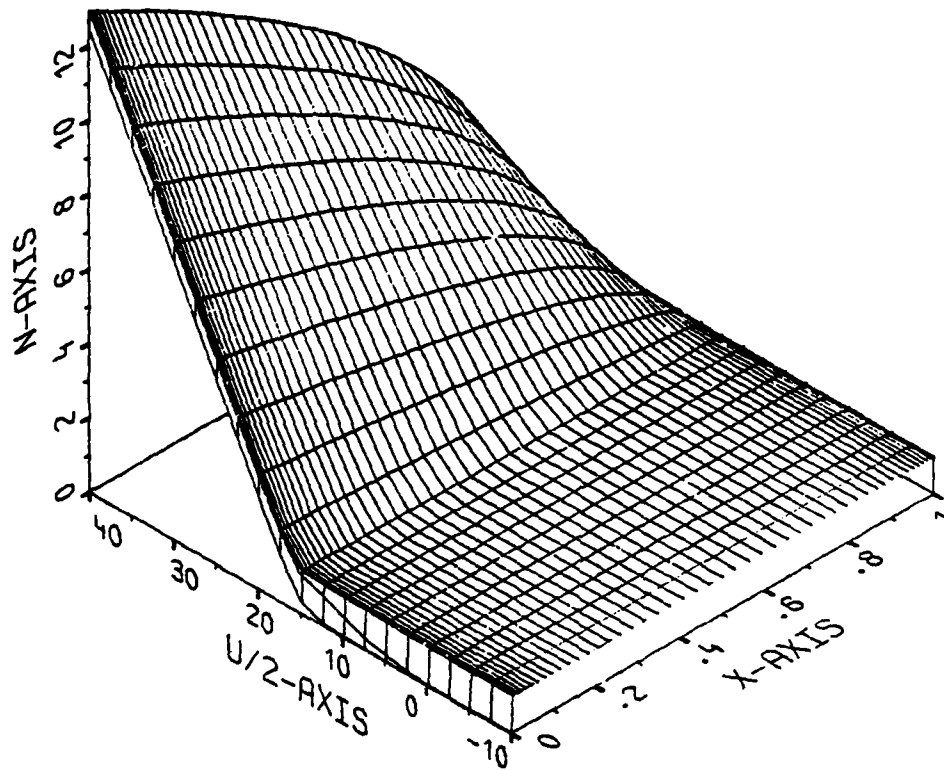

Figure 3.8: Electron Concentration n(x) Parametrized by $\frac{U}{2}$.
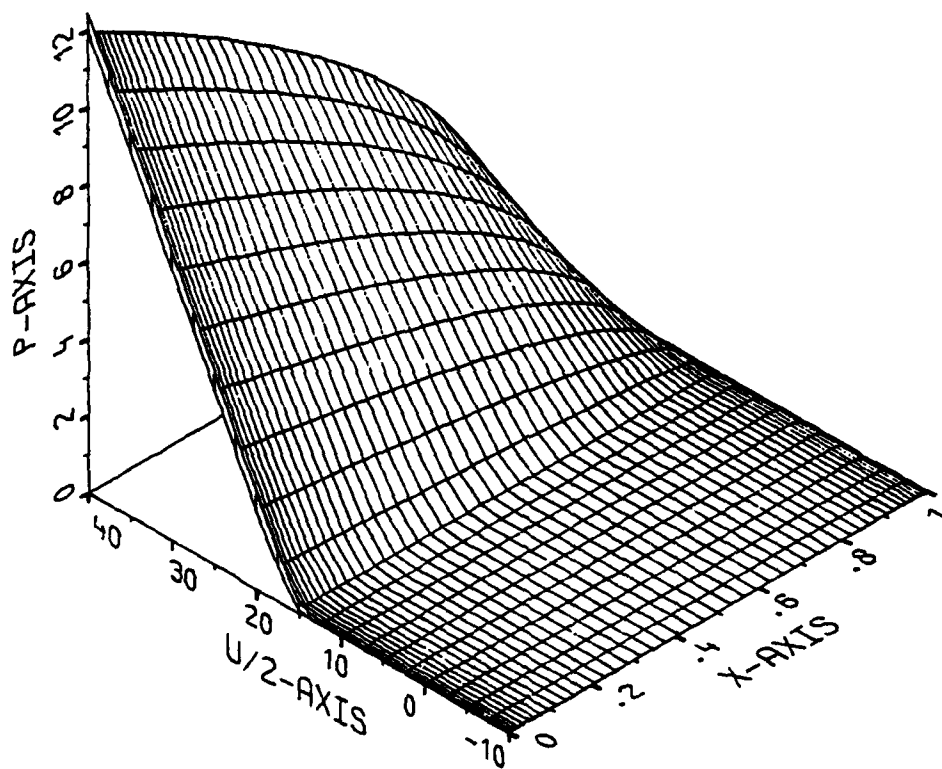
HOLES

Figure 3.9:   Hole Concentration   p(x)   Parametrized by $\frac{U}{2}$.
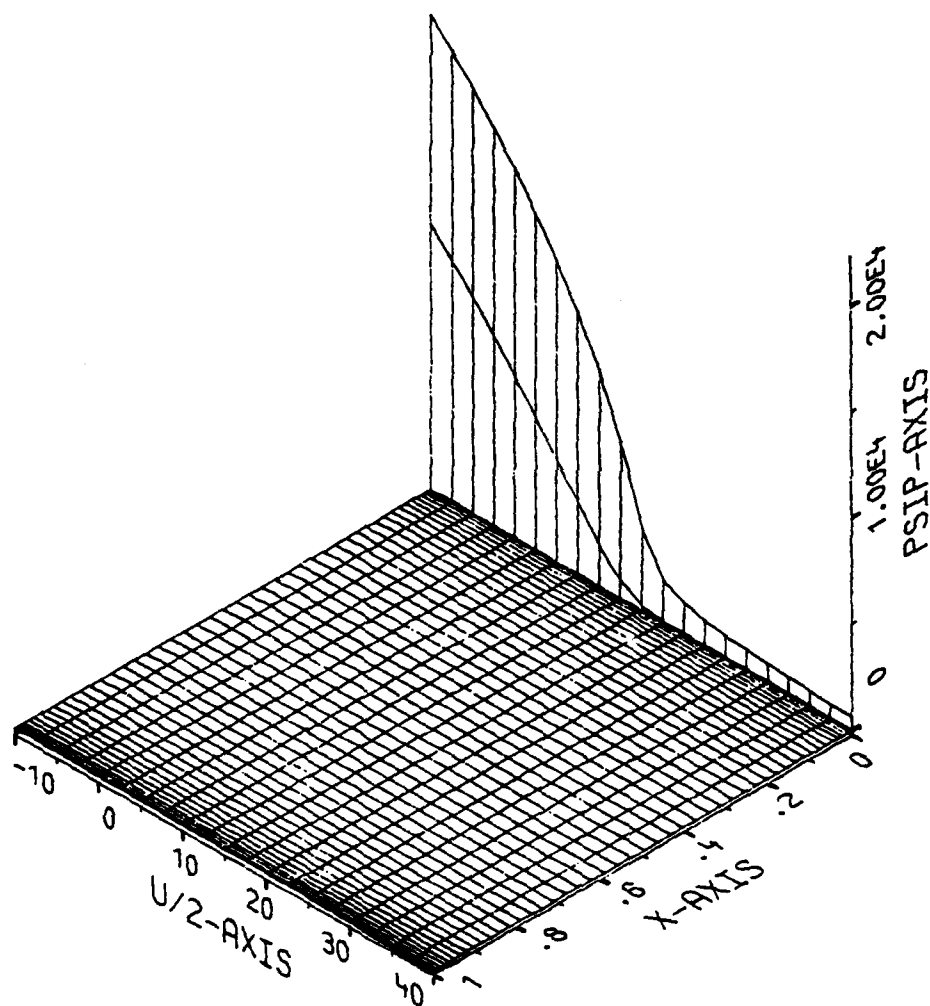
ELECTRIC FIELD

Figure 3.10:   Electric Field $\psi'(x)$ Parametrized by $\frac{U}{2}$.

APPENDIX A: THE LINEARIZED PROBLEM

We shall analyze the linearization of problem (1.1) at the formal approximation of the solution, which has been constructed in Section 2. Thus we consider

$$\lambda^2 \tilde{\psi}'' = \tilde{n} - \tilde{p} + a_1 ,$$

$$\tilde{n}' = (\bar{\psi}' + \frac{1}{\lambda}\dot{\bar{\psi}} + \lambda\bar{\psi}_1' + \dot{\hat{\psi}}_1)\tilde{n} + (\bar{n} + \hat{n} + \lambda\bar{n}_1 + \lambda\hat{n}_1)\tilde{\psi}' + \frac{\tilde{J}}{2} + a_2 ,$$

(A.1)

$$\tilde{p}' = -(\bar{\psi}' + \frac{1}{\lambda}\dot{\bar{\psi}} + \lambda\bar{\psi}_1' + \dot{\hat{\psi}}_1)\tilde{p} - (\bar{p} + \hat{p} + \lambda\bar{p}_1 + \lambda\hat{p}_1)\tilde{\psi}' - \frac{\tilde{J}}{2} + a_3 ,$$

$$\tilde{\psi}(0) = a_4, \quad \tilde{n}(0) - \tilde{p}(0) = a_5, \quad \tilde{\psi}(1) = a_6, \quad \tilde{p}(1) = a_7, \quad \tilde{n}(1) = a_8 ,$$

with $\tilde{y} = (\tilde{\psi}, \tilde{n}, \tilde{p}, \tilde{J}) \in B_{1,\lambda}$ and $a = (a_1, \ldots, a_8) \in B_{2,\lambda}$. We proceed as in Schmeiser and Weiss (1984) and split the interval $[0,1]$ into two parts $[0,x_0]$ and $[x_0,1]$ with x_0 big enough such that the layer terms are small in $[x_0,1]$. We choose $x_0 = 2\lambda\ell n \frac{1}{\lambda}\hat{\psi}(0) = 2\lambda\tau_0$. The constants in conditions (i) and (ii) of Theorem 3.1 can be chosen such that $x_0 < 1$ holds. On the interval $[0,x_0]$ we use τ as independent variable and consider instead of (A.1) the equivalent problem

$$\ddot{\tilde{\psi}}_1 = \tilde{n} - \tilde{p} + b_1 ,$$

(A.2)

$$\dot{\tilde{n}}_1 = \dot{\hat{\psi}}\tilde{n}_1 + (\bar{n}(0) + \hat{n})\dot{\tilde{\psi}}_1 + b_2 + \beta_2 z ,$$

$$\dot{\tilde{p}}_1 = -\dot{\hat{\psi}}\tilde{p}_1 - (\bar{p}(0) + \hat{p})\dot{\tilde{\psi}}_1 + b_3 + \beta_3 z ,$$

for $0 \leqslant \tau \leqslant 2\tau_0$,

$$\lambda^2 \tilde{\psi}_2'' = \tilde{n}_2 - \tilde{p}_2 + b_4 ,$$

(A.3)

$$\tilde{n}_2' = \bar{\psi}'\tilde{n}_2 + (\bar{n} + \lambda\bar{n}_1)\tilde{\psi}_2' + \frac{\tilde{J}}{2} + b_5 + \beta_5 z ,$$

$$\tilde{p}_2' = -\bar{\psi}'\tilde{p}_2 - (\bar{p} + \lambda\bar{p}_1)\tilde{\psi}_2' - \frac{\tilde{J}}{2} + b_6 + \beta_6 z ,$$

for $x_0 \leqslant x \leqslant 1$ and

$$\tilde{\psi}_1(0) = b_7, \quad \tilde{n}_1(0) - \tilde{p}_1(0) = b_8, \quad \tilde{\psi}_2(1) = b_9, \quad \tilde{p}_2(1) = b_{10}, \quad \tilde{n}_2(1) = b_{11},$$

(A.4) $\quad \tilde{\psi}_1(2\tau_0) = \tilde{\psi}_2(x_0) + b_{12}, \quad \dot{\tilde{\psi}}_1(2\tau_0) = \lambda \tilde{\psi}_2'(x_0) + b_{13},$

$$\tilde{n}_1(2\tau_0) = \tilde{n}_2(x_0) + b_{14}, \quad \tilde{p}_1(2\tau_0) = \tilde{p}_2(x_0) + b_{15},$$

where $z = (\tilde{\psi}_1, \tilde{n}_1, \tilde{p}_1, \tilde{\psi}_2, \tilde{n}_2, \tilde{p}_2, \tilde{J}) \in B_{3,\lambda}$ and $b = (b_1, \ldots, b_{15}) \in B_{4,\lambda}$ with
$B_{3,\lambda} = W^{1,\infty}(0,2\tau_0) \times (L^\infty(0,2\tau_0))^2 \times W^{1,\infty}(x_0,1) \times (L^\infty(x_0,1))^2 \times R$ and
$B_{4,\lambda} = L^\infty(0,2\tau_0) \times (L^1(0,2\tau_0))^2 \times L^\infty(x_0,1) \times (L^1(x_0,1))^2 \times R^9$. We introduce $|z|_{3,\lambda}$ on
$B_{3,\lambda}$ and $|b|_{4,\lambda}$ on $B_{4,\lambda}$ defined as the sum of the appropriate norms of the components
of z and b respectively. The b_i in (A.2)-(A.4) are determined by the a_i in (A.1)
and obviously

(A.5) $$|b|_{4,\lambda} \leq |a|_{2,\lambda}$$

holds. An estimate of the perturbations $\beta_i z$ is given in

<u>Lemma A.1</u>: Let $\beta z = (0, \beta_2 z, \beta_3 z, 0, \beta_5 z, \beta_6 z, 0, \ldots, 0) \in B_{4,\lambda}$. Then

(A.6) $$|\beta z|_{4,\lambda} \leq C_1 \lambda \ell n \frac{1}{\lambda} \left[\frac{1}{\sqrt{b-1}} + \delta^4 \sqrt{b} \, (\ell n \sqrt{b})^3 \right] |z|_{3,\lambda}$$

holds.

<u>Proof</u>: Comparing (A.1) to (A.2), (A.3) gives

$$\beta_2 z = (\lambda \bar{\psi}' + \lambda^2 \bar{\psi}_1' + \lambda \dot{\tilde{\psi}}_1)\tilde{n}_1 + (\bar{n} - \bar{n}(0) + \lambda \bar{n}_1 + \lambda \hat{n}_1)\tilde{\psi}_1 + \lambda \frac{\tilde{J}}{2},$$

$$\beta_3 z = -(\lambda \bar{\psi}' + \lambda^2 \bar{\psi}_1' + \lambda \dot{\tilde{\psi}}_1)\tilde{p}_1 - (\bar{p} - \bar{p}(0) + \lambda \bar{p}_1 + \lambda \hat{p}_1)\tilde{\psi}_1 - \lambda \frac{\tilde{J}}{2},$$

$$\beta_5 z = (\frac{1}{\lambda} \dot{\bar{\psi}} + \lambda \bar{\psi}_1' + \dot{\tilde{\psi}}_1)\tilde{n}_2 + (\hat{n} + \lambda \hat{n}_1)\tilde{\psi}_2',$$

$$\beta_6 z = -(\frac{1}{\lambda} \dot{\bar{\psi}} + \lambda \bar{\psi}_1' + \dot{\tilde{\psi}}_1)\tilde{p}_2 - (\hat{p} + \lambda \hat{p}_1)\tilde{\psi}_2'.$$

We use the Lemmas A.d.18 and A.d.20 to obtain

$$\|\beta_2 z\|_{L^1(0,2\tau_0)} <$$

$$< \left[\lambda 2\tau_0 \|\dot{\tilde{\psi}}'\|_{L^\infty(0,x_0)} + \lambda^2 2\tau_0 \|\tilde{\psi}_1'\|_{L^\infty(0,x_0)} + \lambda 2\tau_0 \|\dot{\tilde{\psi}}_1\|_{L^\infty(0,\infty)}\right]\|\tilde{n}_1\|_{L^\infty(0,2\tau_0)} +$$

$$+ \left[2\tau_0 x_0 \|\bar{p}'\|_{L^\infty(0,x_0)} + \lambda 2\tau_0 \|\bar{p}_1\|_{L^\infty(0,x_0)} + \lambda 2\tau_0 \|\hat{n}_1\|_{L^\infty(0,\infty)}\right]\|\dot{\tilde{\psi}}_1\|_{L^\infty(0,2\tau_0)} +$$

$$+ \frac{\lambda}{2}|\tilde{J}| < \mathrm{const}\left[\lambda \ln\frac{1}{\lambda}\sqrt{\ln\sqrt{b}}\left(\frac{1}{b-1} + \delta^4\right) + \right.$$

$$+ \lambda^2 \ln\frac{1}{\lambda}\sqrt{\ln\sqrt{b}}\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ln\sqrt{b}}\right) + \lambda \ln\frac{1}{\lambda}\sqrt{\ln\sqrt{b}}\left(\frac{1}{b} + \delta^4\sqrt{b}\left(\ln\sqrt{b}\right)^{5/2}\right) +$$

$$+ \lambda\left(\ln\frac{1}{\lambda}\right)^2 \ln\sqrt{b}\left(\frac{1}{b-1} + \delta^4\right) + \lambda \ln\frac{1}{\lambda}\sqrt{\ln\sqrt{b}}\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ln\sqrt{b}}\right) +$$

$$+ \lambda \ln\frac{1}{\lambda}\sqrt{\ln\sqrt{b}}\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\left(\ln\sqrt{b}\right)^{5/2}\right) + \lambda\|z\|_{3,\lambda} <$$

$$< \mathrm{const}\,\lambda\,\ln\frac{1}{\lambda}\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\left(\ln\sqrt{b}\right)^3\right)\|z\|_{3,\lambda} \,.$$

Analogously we get

$$\|\beta_3 z\|_{L^1(0,2\tau_0)} < \mathrm{const}\,\lambda\,\ln\frac{1}{\lambda}\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\left(\ln\sqrt{b}\right)^2\right)\|z\|_{3,\lambda} \,.$$

The Lemmas A.d.6, A.d.19 and A.d.20 give

$$\|\beta_5 z\|_{L^1(x_0,1)} < \|\beta_5 z\|_{L^\infty(x_0,1)} <$$

$$< \left[\frac{1}{\lambda}\|\dot{\tilde{\psi}}\|_{L^\infty(2\tau_0,\infty)} + \lambda\|\tilde{\psi}_1'\|_{L^\infty(0,1)} + \|\dot{\tilde{\psi}}_1\|_{L^\infty(2\tau_0,\infty)}\right]\|\tilde{n}_2\|_{L^\infty(x_0,1)} +$$

$$+ \left[\frac{1}{\lambda}\|\hat{n}\|_{L^\infty(2\tau_0,\infty)} + \|\hat{n}_1\|_{L^\infty(2\tau_0,\infty)}\right]\lambda\|\tilde{\psi}_2'\|_{L^\infty(x_0,1)} <$$

$$< \mathrm{const}\left[\lambda\sqrt{\ln\sqrt{b}} + \lambda\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{\ln\sqrt{b}}\right) + \lambda\left(\frac{1}{b}\ln\frac{1}{\lambda} + \delta^4\sqrt{b}\left(\ln\sqrt{b}\right)^{5/2}\right) + \right.$$

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$$+ \lambda \frac{b}{b-1} \ln \sqrt{b} + \lambda \left(\frac{1}{\sqrt{b-1}} \ln \frac{1}{\lambda} + \delta^4 \sqrt{b} \; (\ln \sqrt{b})^{5/2} \right) | z |_{3,\lambda} \; \leqslant$$

$$\leqslant \text{const } \lambda \left(\frac{1}{\sqrt{b-1}} \ln \frac{1}{\lambda} + \delta^4 \sqrt{b} \; (\ln \sqrt{b})^{5/2} \right) | z |_{3,\lambda} \; .$$

Analogously we obtain

$$| B_6 z |_{L^1(x_0,1)} \leqslant \text{const } \lambda \left(\frac{1}{\sqrt{b-1}} \ln \frac{1}{\lambda} + \delta^4 \sqrt{b} \; (\ln \sqrt{b})^{5/2} \right) | z |_{3,\lambda} \; .$$

For the application of a contraction mapping argument to (A.2)-(A.4) we analyze the unperturbed problem

$$\ddot{\tilde{\psi}}_1 = \tilde{n} - \tilde{p} + d_1 \; ,$$

(A.7)
$$\dot{\tilde{n}}_1 = \dot{\hat{\psi}} \tilde{n}_1 + (\bar{n}(0) + \hat{n}) \dot{\tilde{\psi}}_1 + d_2 \; ,$$

$$\dot{\tilde{p}}_1 = -\dot{\hat{\psi}} \tilde{p}_1 - (\bar{p}(0) + \hat{p}) \dot{\tilde{\psi}}_1 + d_3 \; ,$$

for $0 \leqslant \tau \leqslant 2\tau_0$,

$$\lambda^2 \tilde{\psi}_2'' = \tilde{n}_2 - \tilde{p}_2 + d_4 \; ,$$

(A.8)
$$\tilde{n}_2' = \bar{\psi}' \tilde{n}_2 + (\bar{n} + \lambda \bar{n}_1) \tilde{\psi}_2' + \frac{\tilde{J}}{2} + d_5 \; ,$$

$$\tilde{p}_2' = -\bar{\psi}' \tilde{p}_2 - (\bar{p} + \lambda \bar{p}_1) \tilde{\psi}_2' - \frac{\tilde{J}}{2} + d_6 \; ,$$

for $x_0 \leqslant x \leqslant 1$ and

$$\tilde{\psi}_1(0) = d_7, \quad \tilde{n}_1(0) - \tilde{p}_1(0) = d_8, \quad \tilde{\psi}_2(1) = d_9, \quad \tilde{p}_2(1) = d_{10}, \quad \tilde{n}_2(1) = d_{11} \; ,$$

(A.9)
$$\tilde{\psi}_1(2\tau_0) = \tilde{\psi}_2(x_0) + d_{12}, \quad \dot{\tilde{\psi}}_1(2\tau_0) = \lambda \tilde{\psi}_2'(x_0) + d_{13} \; ,$$

$$\tilde{n}_1(2\tau_0) = \tilde{n}_2(x_0) + d_{14}, \quad \tilde{p}_1(2\tau_0) = \tilde{p}_2(x_0) + d_{15} \; ,$$

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where we introduced the new inhomogeneity $d = (d_1, \ldots, d_{15}) \in B_{4,\lambda}$. In order to identify slow and fast variables in (A.8) we apply the following transformation:

(A.10)
$$\tilde{\psi}_2 = t - u$$
$$\lambda\tilde{\psi}_2' = \sqrt{2\bar{p} + 1}\, v$$
$$\tilde{n}_2 = -(\bar{p} + 1)u + w$$
$$\tilde{p}_2 = \bar{p}u + w$$

Substitution into (A.8) yields

$$\lambda u' = -\sqrt{2\bar{p} + 1}\, v - \frac{\lambda}{2\bar{p} + 1}\left(2\bar{p}_1\sqrt{2\bar{p} + 1}\, v + 2\tilde{\psi}'w + \tilde{J}\right) + \frac{\lambda}{2\bar{p} + 1}(d_6 - d_5)\,,$$

$$\lambda v' = -\sqrt{2\bar{p} + 1}\, u + \frac{\lambda\bar{J}}{2(2\bar{p} + 1)^2}\, v + \frac{1}{\sqrt{2\bar{p} + 1}}\, d_4\,,$$

(A.11)

$$w' = \frac{\bar{J}}{2}\, u - \frac{\bar{p}_1}{\sqrt{2\bar{p} + 1}}\, v + \frac{\bar{J}}{(2\bar{p} + 1)^2}\, w - \frac{1}{2(2\bar{p} + 1)}\, \tilde{J} + \frac{\bar{p}d_5 + (\bar{p} + 1)d_6}{2\bar{p} + 1}\,,$$

$$t' = -\frac{2\bar{p}_1}{\sqrt{2\bar{p} + 1}}\, v + \frac{2\bar{J}}{(2\bar{p} + 1)^2}\, w - \frac{1}{2\bar{p} + 1}\, \tilde{J} + \frac{d_6 - d_5}{2\bar{p} + 1}\,.$$

As in Schmeiser and Weiss (1984) we apply a decoupling transformation:

(A.12)
$$\bar{w} = w - \frac{\lambda\bar{p}_1}{2\bar{p} + 1}\, u + \frac{\lambda\bar{J}}{2\sqrt{2\bar{p} + 1}}\, v\,,$$
$$\bar{t} = t - \frac{2\lambda\bar{p}_1}{2\bar{p} + 1}\, u\,.$$

With these transformations (A.7)-(A.9) become

$$\ddot{\tilde{\psi}}_1 = \tilde{n}_1 - \tilde{p}_1 + e_1 \, ,$$

(A.13)
$$\dot{\tilde{n}}_1 = \dot{\hat{\psi}}\tilde{n}_1 + (\bar{n}(0) + \hat{n})\dot{\tilde{\psi}}_1 + e_2 \, ,$$

$$\dot{\tilde{p}}_1 = \dot{\hat{\psi}}\tilde{p}_1 - (\bar{p}(0) + \hat{p})\dot{\tilde{\psi}}_1 + e_3 \, ,$$

for $0 < \tau < 2\tau_0$,

$$\lambda u' = -\sqrt{2\bar{p} + 1} \, v + e_4 + E_4 \bar{z}$$

(A.14)
$$\lambda v' = -\sqrt{2\bar{p} + 1} \, u + e_5 + E_5 \bar{z}$$

$$\bar{w}' = \frac{\bar{J}}{(2\bar{p} + 1)^2} \, \bar{w} - \frac{1}{2(2\bar{p} + 1)} \tilde{J} + e_6 + E_6 \bar{z}$$

(A.15)
$$\bar{t}' = \frac{2\bar{J}}{(2\bar{p} + 1)^2} \, \bar{w} - \frac{1}{2\bar{p} + 1} \tilde{J} + e_7 + E_7 \bar{z}$$

for $x_0 < x < 1$ and

$$\tilde{\psi}_1(0) = e_8, \quad \tilde{n}_1(0) - \tilde{p}_1(0) = e_9, \quad u(1) = e_{10} \, ,$$

$$\bar{w}(1) = e_{11} + E_{11}\bar{z}(1), \quad E(1) = e_{12} \, ,$$

$$\tilde{\psi}_1(2\tau_0) = \bar{t}(x_0) - u(x_0) + e_{13} + E_{13}\bar{z}(x_0) \, ,$$

(A.16)
$$\dot{\tilde{\psi}}_1(2\tau_0) = \sqrt{2\bar{p}(x_0) + 1} \, v(x_0) + e_{14} \, ,$$

$$\tilde{n}_1(2\tau_0) = -(\bar{p}(x_0) + 1)u(x_0) + \bar{w}(x_0) + e_{15} + E_{15}\bar{z}(x_0)$$

$$\tilde{p}_1(2\tau_0) = \bar{p}(x_0)u(x_0) + \bar{w}(x_0) + e_{16} + E_{16}\bar{z}(x_0) \, ,$$

where $\bar{z} = (\tilde{\psi}_1, \tilde{n}_1, \tilde{p}_1, u, v, \bar{w}, \bar{t}, \tilde{J}) \in B_{5,\lambda}$ and $e = (e_1, \ldots, e_{16}) \in B_{6,\lambda}$ with $B_{5,\lambda} = W^{1,\infty}(0, 2\tau_0) \times (L^\infty(0, 2\tau_0))^2 \times (L^\infty(x_0, 1))^4 \times R$ and $B_{6,\lambda} = L^\infty(0, 2\tau_0) \times (L^1(0, 2\tau_0))^2 \times L^1(x_0, 1) \times L^\infty(x_0, 1) \times (L^1(x_0, 1))^2 \times R^9$. As norm $\|\cdot\|_{5,\lambda}$ on $B_{5,\lambda}$ we take the sum of the natural norms of the components of $\bar{z}$, whereas in the definition of

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$\|e\|_{6,\lambda}$ on $B_{6,\lambda}$ we replace $\|e_4\|_{L^1(x_0,1)}$ by $\lambda^{-1}\|e_4\|_{L^1(x_0,1)}$. The following two lemmas give estimates for e in terms of d and for the perturbations $E_i\bar{z}$ in terms of $\bar{z}$:

Lemma A.2: The estimate

$$\|e\|_{6,\lambda} < \text{const} \frac{b}{b-1} \|d\|_{4,\lambda}$$

holds.

Proof: Comparing (A.7)-(A.9) to (A.13)-(A.16) yields

$$e_i = d_i \quad \text{for} \quad i = 1,2,3 ,$$

$$e_4 = \frac{\lambda}{2\bar{p}+1}(d_6 - d_5), \quad e_5 = \frac{1}{\sqrt{2\bar{p}+1}} d_4 ,$$

$$e_6 = \frac{\bar{J}}{2(2\bar{p}+1)} d_4 + \left(\bar{p} + \frac{\lambda\bar{p}_1}{2\bar{p}+1}\right)\frac{d_5}{2\bar{p}+1} + \left(\bar{p} + 1 - \frac{\lambda\bar{p}_1}{2\bar{p}+1}\right)\frac{d_6}{2\bar{p}+1} ,$$

$$e_7 = \left(1 - \frac{2\lambda\bar{p}_1}{2\bar{p}+1}\right)\frac{d_6 - d_5}{2\bar{p}+1}, \quad e_{10} = \frac{d_{10} - d_{11}}{2p_1+1} ,$$

$$e_{11} = \frac{(p_1+1)d_{10} + p_1 d_{11}}{2p_1+1}, \quad e_{12} = d_9 + \frac{d_{10} - d_{11}}{2p_1+1} ,$$

$$e_j = d_{j-1} \quad \text{for} \quad j = 8,9,13,\dots,16 .$$

Obviously

$$\|e_6\|_{L^1(x_0,1)} < \text{const}\left[|\bar{J}| \left\|\frac{1}{2\bar{p}+1}\right\|_{L^1(x_0,1)} \|d_4\|_{L^\infty(x_0,1)} + \|d_5\|_{L^1(x_0,1)} + \|d_6\|_{L^1(x_0,1)}\right]$$

The estimate (A.d. 11) implies

$$\left\|\frac{1}{2\bar{p}+1}\right\|_{L^1(x_0,1)} < \left\|\frac{b-1}{(b+1)\sqrt{1-x}}\right\|_{L^1(0,1)} = 2\frac{b-1}{b+1} .$$

With $|\bar{J}| < \text{const}\left(\frac{b+1}{b-1}\right)^2$ we obtain

$$\|e_6\|_{L^1(x_0,1)} < \text{const} \frac{b}{b-1} \|d\|_{4,\lambda}$$

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Using the nonnegativity of $\bar{p}$ in estimating the remaining components of e, the result of Lemma A.2 follows.

Lemma A.3: Let $E\bar{z} = (0,0,0,E_4\bar{z},\ldots,E_7\bar{z},0,0,0,E_{11}\bar{z},0,E_{13}\bar{z},0,E_{15}\bar{z},E_{16}\bar{z}) \in B^*_{6,\lambda}$, where $B^*_{6,\lambda}$ is as $B_{6,\lambda}$, but with the fourth component $L^\infty(x_0,1)$ instead of $L^1(x_0,1)$. The norm $|e|^*_{6,\lambda}$ on $B^*_{6,\lambda}$ is defined by replacing $\lambda^{-1}|e_4|_{L^1(x_0,1)}$ in $|e|_{6,\lambda}$ by $|e_4|_{L^\infty(x_0,1)}$. Then

$$|E\bar{z}|^*_{6,\lambda} \leq C_2\lambda\left(\frac{1}{(b-1)^2} + \delta\sqrt[4]{\ln\sqrt{b}}\right)|\bar{z}|_{5,\lambda}$$

holds.

Proof: Using (A.11) and Lemma A.d.20 gives

$$|E_4\bar{z}|_{L^\infty(x_0,1)} \leq \text{const } \lambda\left(\left|\frac{\bar{p}_1}{\sqrt{2\bar{p}+1}}\right|_{L^\infty(x_0,1)} + |\bar{\psi}'|_{L^\infty(x_0,1)} + 1\right)|\bar{z}|_{5,\lambda}$$

$$\leq \text{const } \lambda\left(\frac{\sqrt{b-1}}{b} + \delta\sqrt[4]{\ln\sqrt{b}} + \frac{b}{(b-1)^2} + \delta^4\right)|\bar{z}|_{5,\lambda}$$

$$\leq \text{const } \lambda\left(\frac{b}{(b-1)^2} + \delta\sqrt[4]{\ln\sqrt{b}}\right)|\bar{z}|_{5,\lambda}$$

$$|E_5\bar{z}|_{L^\infty(x_0,1)} \leq \text{const } \lambda|\bar{J}|\left|\frac{1}{(2\bar{p}+1)^2}\right|_{L^\infty(x_0,1)}|\bar{z}|_{5,\lambda}$$

$$\leq \text{const } \lambda\left(\frac{b}{(b-1)^2} + \delta^4\right)|\bar{z}|_{5,\lambda}$$

Elementary calculations show

$$E_6\bar{z} = \lambda\left[\left(\frac{\bar{J}2\bar{p}_1}{(2\bar{p}+1)^3} - \frac{\bar{p}_1'}{2\bar{p}+1}\right)u + \left(\frac{2\bar{p}_1^2}{(2\bar{p}+1)^{3/2}} + \frac{\bar{J}^2}{(2\bar{p}+1)^{5/2}}\right)v - \frac{2\bar{J}\bar{p}_1}{(2\bar{p}+1)^3}w + \frac{\bar{p}_1}{(2\bar{p}+1)^2}\bar{J}\right]$$

and

$$E_7\bar{z} = 2E_6\bar{z}.$$

From (2.8) we get

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$$\left|\frac{1}{(2\bar{p}+1)^{5/2}}\right|_{L^1(0,1)} < \text{const}\left(\frac{b-1}{b+1}\right)^2 .$$

The Lemma A.d.20 and (A.d.12) give

$$|E_6\bar{z}|_{L^1(x_0,1)} , \; |E_7\bar{z}|_{L^1(x_0,1)} < \text{const } \lambda\left[\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ell n\sqrt{b}}\right)\left(\frac{b}{(b-1)^2} + \delta^4\right) + \right.$$

$$+ \frac{1}{\sqrt{b-1}} + \delta^4\sqrt{\ell n\sqrt{b}} + \frac{b-1}{b^2} + \delta^8\ell n\sqrt{b} + \left(\frac{b^2}{(b-1)^4} + \delta^8\right)\left(\frac{b-1}{b+1}\right)^2 +$$

$$\left. + \left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ell n\sqrt{b}}\right)\left(\frac{b}{(b-1)^2} + \delta^4\right) + \frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ell n\sqrt{b}}\right]|\bar{z}|_{5,\lambda} <$$

$$< \text{const } \lambda\left(\frac{b}{(b-1)^2} + \delta^4\sqrt{\ell n\sqrt{b}}\right)|\bar{z}|_{5,\lambda}$$

The perturbations in the boundary conditions contain the terms $\dfrac{\lambda\bar{J}}{2\sqrt{2\bar{p}+1}}\,v$ and $\dfrac{\lambda\bar{p}_1}{2\bar{p}+1}\,u$

at $x = x_0$ and $x = 1$. Estimates follow from Lemma A.d.20.

In order to be able to apply a contraction mapping argument to (A.13)-(A.16) we analyze the unperturbed problem

(A.17a)
$$\ddot{\tilde{\psi}}_1 = \tilde{n}_1 - \tilde{p}_1 + f_1 ,$$

(A.17b)
$$\dot{\tilde{n}}_1 = \dot{\hat{\psi}}\tilde{n}_1 + (\bar{n}(0) + \hat{n})\dot{\tilde{\psi}}_1 + f_2 ,$$

(A.17c)
$$\dot{\tilde{p}}_1 = -\dot{\hat{\psi}}\tilde{p}_1 - (\bar{p}(0) + \hat{p})\dot{\tilde{\psi}}_1 + f_3 ,$$

for $0 \leq \tau \leq 2\tau_0$,

(A.18)
$$\lambda u' = -\sqrt{2\bar{p}+1}\,v + f_4 ,$$

$$\lambda v' = -\sqrt{2\bar{p}+1}\,u + f_5 ,$$

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$$\bar{w}' = \frac{\bar{J}}{(2\bar{p}+1)^2}\,\bar{w} - \frac{1}{2(2\bar{p}+1)}\,\tilde{J} + f_6 \; ,$$

$$\bar{t}' = \frac{2\bar{J}}{(2\bar{p}+1)^2}\,\bar{w} - \frac{1}{2\bar{p}+1}\,\tilde{J} + f_7 \; ,$$

for $x_0 \leqslant x \leqslant 1$ and

$$\tilde{\psi}_1(0) = f_8, \quad \tilde{n}_1(0) - \tilde{p}_1(0) = f_9, \quad u(1) = f_{10}, \quad \bar{w}(1) = f_{11}, \quad \bar{t}(1) = f_{12} \; ,$$

$$\tilde{\psi}_1(2\tau_0) = \bar{t}(x_0) - u(x_0) + f_{13}, \quad \dot{\tilde{\psi}}_1(2\tau_0) = \sqrt{2\bar{p}(x_0)+1}\; v(x_0) + f_{14} \, ,$$

(A.20)

$$\tilde{n}_1(2\tau_0) = -(\bar{p}(x_0)+1)u(x_0) + \bar{w}(x_0) + f_{15} \; ,$$

$$\tilde{p}_1(2\tau_0) = \bar{p}(x_0)u(x_0) + \bar{w}(x_0) + f_{16} \; ,$$

where $f = (f_1,\ldots,f_{16})$ is from $B_{6,\lambda}$ or $B_{6,\lambda}^*$. The differential equations in problem (A.17)-(A.20) contain the linearization of the layer equations (2.4) in (A.17), a standard linear singularly perturbed system in (A.18) and the linearization of the reduced equations (2.3) in (A.19). These systems are analyzed in the paragraphs a), b) and c) of this appendix. We then substitute (A.a.1), (A.b.3) and (A.c.1) into the boundary conditions (A.20). We replace τ_0 by ∞, x_0 by 0 and $\nu_h(x_0)$ and $\mu_h(1)$ by 0 in the coefficient matrix of the resulting linear system for the c_i's. Thus, we solve a system of the form $Ac = \eta$ instead of $(A + F)c = \eta$, where

$$|F|_\infty \leqslant c_3\left(\lambda\sqrt{\frac{b+1}{b-1}}\; (\ell n\sqrt{b})^{3/2} + \exp\left(\frac{x_0-1}{\lambda}\right) + \lambda\ell n\,\frac{1}{\lambda}\sqrt{\ell n\sqrt{b}}\right) \; .$$

This estimate follows from the Lemmas A.a.3, A.b.1, Taylor's theorem and the estimates in the proof of Lemma A.d.20. The system with $F = 0$ is given by

$$c_2 = f_8 \, ,$$

$$c_2 \frac{2\sqrt{b}}{b-1} + c_3 \frac{1}{\sqrt{b}} - c_4 = f_9 - \tilde{n}_q(0) \, ,$$

$$c_6 = f_{10} - \mu_p(1) \, ,$$

$$c_7 \frac{1}{2p_1 + 1} - c_9 \frac{1}{2(2p_1 + 1)} = f_{11} \, ,$$

$$c_7 \frac{2}{2p_1 + 1} + c_8 - c_9 \frac{1}{2p_1 + 1} = f_{12} - \bar{t}_p(1) \, ,$$

$$\text{(A.21)} \qquad c_1 \frac{1}{2} \sqrt{\frac{b-1}{b+1}} - c_3 \frac{b-1}{b+1} + c_4 \frac{b-1}{\sqrt{b}(b+1)} + c_5 - c_7 \frac{2(b-1)}{b+1} - c_8 =$$

$$= f_{13} - \tilde{\psi}_p(2\tau_0) + 2\bar{w}_p(x_0) - \nu_p(x_0) \, ,$$

$$c_1 \frac{1}{2} - c_5 \sqrt{\frac{b+1}{b-1}} = f_{14} - \dot{\tilde{\psi}}_p(2\tau_0) - \nu_p(x_0) \sqrt{\frac{b+1}{b-1}} \, ,$$

$$c_1 \frac{b}{2\sqrt{b^2 - 1}} + c_3 \frac{1}{b+1} - c_4 \frac{\sqrt{b}}{b+1} + c_5 \frac{b}{b-1} - c_7 \frac{b-1}{b+1} =$$

$$= f_{15} - \tilde{n}_p(2\tau_0) - \nu_p(x_0) \frac{b}{b-1} + \bar{w}_p(x_0) \, ,$$

$$-c_1 \frac{1}{2\sqrt{b^2 - 1}} + c_3 \frac{1}{b+1} + c_4 \frac{b+2}{\sqrt{b}(b+1)} - c_5 \frac{1}{b-1} - c_7 \frac{b-1}{b+1} =$$

$$= f_{16} - \tilde{p}_p(2\tau_0) + \nu_p(x_0) \frac{1}{b-1} + \bar{w}_p(x_0) \, .$$

Lengthy calculations show that the coefficient matrix satisfies

$$|A^{-1}|_\infty \leq c_4 \frac{b}{b-1} \, .$$

With the constants in Theorem 3.1 chosen appropriately, the estimate

$$|A^{-1}|_\infty |F|_\infty \leq \frac{1}{2}$$

holds, which implies the nonsingularity of $A + F$ and

$$|(A + F)^{-1}|_\infty \leq 2c_4 \frac{b}{b-1}$$

Using the estimates on the particular solutions in lemma A.a.1, in (A.b.4) and (A.c.2) we

obtain an estimate of the righthand side η of (A.21):

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$$|n|_\infty \leq \text{const}\left(\frac{1}{b-1} + \ln\sqrt{b}\right)|f|_{6,\lambda}$$

This implies for $c = (c_1,\ldots,c_9)$:

(A.22)
$$|c|_\infty \leq \text{const}\left(\frac{b}{(b-1)^2} + \ln\sqrt{b}\right)|f|_{6,\lambda}$$

Using (A.22), the estimates (A.b.4), (A.c.2) and Lemma A.a.1 we obtain

$$|\bar{z}|_{5,\lambda} \leq \text{const}\left(\frac{1}{(b-1)^3} + (\ln\sqrt{b})^2\right)|f|_{6,\lambda}$$

The difference between $|\cdot|_{6,\lambda}$ and $|\cdot|_{6,\lambda}^*$ only appears in the fourth component. Using

(A.b.5) we immediately obtain

(A.23)
$$|\bar{z}|_{5,\lambda} \leq c_5\left(\frac{1}{(b-1)^3} + (\ln\sqrt{b})^2\right)|f|_{6,\lambda}^*$$

Lemma A.3 and (A.23) imply the condition

(A.24)
$$c_2 c_5 \lambda\left(\frac{1}{(b-1)^5} + \delta^4(\ln\sqrt{b})^{5/2}\right) \leq \frac{1}{2}$$

for the applicability of a contraction mapping argument. With the constants in Theorem 3.1

chosen appropriately (A.24) is satisfied and the solvability of problem (A.13)-(A.16)

follows. Furthermore we obtain the estimate

$$|\bar{z}|_{5,\lambda} \leq \text{const}\left(\frac{1}{(b-1)^3} + (\ln\sqrt{b})^2\right)|e|_{6,\lambda} ,$$

and from Lemma A.2

(A.25)
$$|\bar{z}|_{5,\lambda} \leq \text{const}\left(\frac{1}{(b-1)^4} + (\ln\sqrt{b})^2\right)|d|_{4,\lambda} .$$

The transformations (A.10) and (A.12) give

$$|z|_{3,\lambda} \leq \text{const} \frac{b}{b-1} |\bar{z}|_{5,\lambda} .$$

Thus, (A.25) implies

(A.26)
$$|z|_{3,\lambda} \leq c_6\left(\frac{1}{(b-1)^5} + (\ln\sqrt{b})^2\right)|d|_{4,\lambda} .$$

If the condition

$$c_1 c_6 \lambda \ln\frac{1}{\lambda}\left(\frac{1}{(b-1)^{11/2}} + \delta^4\sqrt{b}(\ln\sqrt{b})^5\right) \leq \frac{1}{2}$$

is satisfied (which again is guaranteed by the assumptions in Theorem 3.1) we can apply a contraction mapping argument to obtain solvability of problem (A.2)-(A.4) and the estimate

$$(A.27) \qquad |z|_{3,\lambda} < const\Big(\frac{1}{(b-1)^5} + (\ln\sqrt{b})^2\Big)|b|_{4,\lambda} \ .$$

Thus, problem (A.1) also has a unique solution $\tilde{y}$. The differential equations in (A.1) can be used to show

$$\tilde{y} \in B_{1,\lambda}$$

and the estimate

$$(A.28) \qquad |\tilde{y}|_{1,\lambda} < const \frac{b}{b-1} |z|_{3,\lambda} \ .$$

Finally (A.5), (A.27) and (A.28) imply the validity of the

__Lemma A.4:__ Problem A.1 has a unique solution $\tilde{y}$ and the estimate

$$|\tilde{y}|_{1,\lambda} < const\Big(\frac{1}{(b-1)^6} + (\ln\sqrt{b})^2\Big)|a|_{2,\lambda}$$

holds.

a) __The linearized layer equations.__

We consider (A.17). Integration of (A.17b,c) and using (2.6), (2.8) yield

$$(A.a.1) \qquad \tilde{n}_1 = c_3 e^{\hat{\psi}} + \frac{b}{b-1} e^{\hat{\psi}}\tilde{\psi}_1 + \tilde{n}_q \ ,$$

$$\tilde{p}_1 = \frac{c_4}{\sqrt{b}} e^{-\hat{\psi}} - \frac{1}{b-1} e^{-\hat{\psi}}\tilde{\psi}_1 + \tilde{p}_q \ ,$$

where c_3, c_4 are constants and

$$\tilde{n}_q(\tau) = \int_{2\tau_0}^{\tau} e^{\hat{\psi}(\tau)-\hat{\psi}(s)} f_2(s)ds \ ,$$

$$\tilde{p}_q(\tau) = \int_{0}^{\tau} e^{-\hat{\psi}(\tau)+\hat{\psi}(s)} f_3(s)ds \ .$$

The monotonicity of $\hat{\psi}$ immediately implies the estimates

$$(A.a.2) \qquad |\tilde{n}_q|_{L^\infty(0,2\tau_0)} < |f_2|_{L^1(0,2\tau_0)} \ , \quad |\tilde{p}_q|_{L^\infty(0,2\tau_0)} < |f_3|_{L^1(0,2\tau_0)} \ .$$

Substituting (A.a.1) into (A.17a) gives

$$\ddot{\tilde{\psi}}_1 = \left(\frac{b}{b-1}e^{\hat{\psi}} + \frac{1}{b-1}e^{-\hat{\psi}}\right)\tilde{\psi}_1 + c_3 e^{\hat{\psi}} - \frac{c_4}{\sqrt{b}}e^{-\hat{\psi}} + \tilde{n}_q - \tilde{p}_q + f_1 \qquad \text{(A.a.3)}$$

The homogeneous part of (A.a.3) has the linearly independent solutions $\dot{\hat{\psi}}(\tau)$ and $\dot{\hat{\psi}}(\tau)\int_0^\tau \dot{\hat{\psi}}^{-2}(s)ds$. For the construction of a particular solution of the inhomogeneous equation we use the results of Fife (1974) and get

$$\tilde{\psi}_1(\tau) = c_1\dot{\hat{\psi}}(2\tau_0)\dot{\hat{\psi}}(\tau)\int_0^\tau \dot{\hat{\psi}}^{-2}(s)ds + c_2\dot{\hat{\psi}}(\tau)\dot{\hat{\psi}}^{-1}(0) -$$

$$- \dot{\hat{\psi}}(\tau)\int_0^\tau \dot{\hat{\psi}}^{-2}(s)\left\{\int_s^\infty \dot{\hat{\psi}}(u)\left[c_3 e^{\hat{\psi}(u)} - \frac{c_4}{\sqrt{b}}e^{-\hat{\psi}(u)}\right]du\right.$$

$$\left. + \int_s^{2\tau_0} \dot{\hat{\psi}}(u)[\tilde{n}_q(u) - \tilde{p}_q(u) + f_1(u)]du\right\}ds$$

(A.a.4a)

$$= c_1\dot{\hat{\psi}}(2\tau_0)\dot{\hat{\psi}}(\tau)\int_0^\tau \dot{\hat{\psi}}^{-2}(s)ds + c_2\dot{\hat{\psi}}(\tau)\dot{\hat{\psi}}^{-1}(0)$$

$$- c_3\dot{\hat{\psi}}(\tau)\int_0^\tau (1 - e^{\hat{\psi}(s)})\dot{\hat{\psi}}^{-2}(s)ds$$

$$- c_4\dot{\hat{\psi}}(\tau)\int_0^\tau (e^{-\hat{\psi}(s)} - 1)\frac{1}{\sqrt{b}}\dot{\hat{\psi}}^{-2}(s)ds + \tilde{\psi}_p(\tau) ,$$

where $\tilde{\psi}_p(\tau) = -\dot{\hat{\psi}}(\tau)\int_0^\tau \dot{\hat{\psi}}^{-2}(s)\int_s^{2\tau_0}\dot{\hat{\psi}}(u)[\tilde{n}_q(u) - \tilde{p}_q(u) + f_1(u)]du\,ds$. Differentiation of (A.a.4a) gives

$$\text{(A.a.4b)} \quad \dot{\tilde{\psi}}_1(\tau) = c_1\dot{\hat{\psi}}(2\tau_0)\left[(\hat{n}(\tau) - \hat{p}(\tau))\int_0^\tau \dot{\hat{\psi}}^{-2}(s)ds + \dot{\hat{\psi}}^{-1}(\tau)\right] + c_2(\hat{n}(\tau) - \hat{p}(\tau))\dot{\hat{\psi}}^{-1}(0)$$

$$- c_3\left[(\hat{n}(\tau) - \hat{p}(\tau))\int_0^\tau (1 - e^{\hat{\psi}(s)})\dot{\hat{\psi}}^{-2}(s)ds + (1 - e^{\hat{\psi}(\tau)})\dot{\hat{\psi}}^{-1}(\tau)\right] -$$

$$- c_4 \left[(\hat{n}(\tau) - \hat{p}(\tau)) \int_0^\tau (e^{-\hat{\psi}(s)} - 1) \frac{1}{\sqrt{b}} \dot{\hat{\psi}}^{-2}(s) ds + (e^{-\hat{\psi}(\tau)} - 1) \frac{1}{\sqrt{b}} \dot{\hat{\psi}}^{-1}(\tau) \right]$$

$$+ \dot{\tilde{\psi}}_p(\tau) \ ,$$

The estimates of the solution are collected in

<u>Lemma A.a.1</u>:

a) $\|\dot{\tilde{\psi}}_1\|_{L^\infty(0,2\tau_0)} \leq \mathrm{const} (\sqrt{\ln\sqrt{b}} \ |c_1| + |c_2| + \ln\sqrt{b} \ |c_3| + \ln\sqrt{b} \ |c_4| + \ln\sqrt{b} \ \|f\|_{6,\lambda})$,

b) $\|\dot{\tilde{\psi}}_1\|_{L^\infty(0,2\tau_0)} \leq \mathrm{const} \left((1 + \sqrt{\ln\sqrt{b}}) \ |c_1| + \sqrt{\frac{b}{b-1}} \ |c_2| + \ln\sqrt{b} \ |c_3| \right.$

$$\left. + \ln\sqrt{b} \ |c_4| + (1 + \ln\sqrt{b}) \|f\|_{6,\lambda} \right) \ ,$$

c) $\|\tilde{n}_1\|_{L^\infty(0,2\tau_0)} \leq \mathrm{const} \left((\frac{1}{\sqrt{b-1}} + \sqrt{\ln\sqrt{b}}) \ |c_1| + \frac{b}{b-1} \ |c_2| + \right.$

$$\left. + (1 + \ln\sqrt{b}) \ |c_3| + (1 + \ln\sqrt{b}) \ |c_4| + (1 + \ln\sqrt{b}) \ \|f\|_{6,\lambda} \right) \ ,$$

d) $\|\tilde{p}_1\|_{L^\infty(0,2\tau_0)} \leq \mathrm{const} \left(\sqrt{\frac{b}{b-1}} \ |c_1| + \frac{\sqrt{b}}{b-1} \ |c_2| + |c_3| + |c_4| + \|f\|_{6,\lambda} \right) \ .$

<u>Proof</u>: The results are obtained by using the Lemmas A.d.4, A.d.5, A.d.8, A.d.10, A.d.11 and A.d.12 and the estimates (A.a.2).

With $\tilde{n}_p(\tau) = \frac{b}{b-1} e^{\hat{\psi}(\tau)} \dot{\tilde{\psi}}_p(\tau) + \tilde{n}_q(\tau)$ and $\tilde{p}_p(\tau) = -\frac{1}{b-1} e^{-\hat{\psi}(\tau)} \dot{\tilde{\psi}}_p(\tau) + \tilde{p}_q(\tau)$ we define

$$\tilde{\psi}_h(\tau) = \tilde{\psi}_1(\tau) - \tilde{\psi}_p(\tau) \ ,$$
$$\tilde{n}_h(\tau) = \tilde{n}_1(\tau) - \tilde{n}_p(\tau) \ ,$$
$$\tilde{p}_h(\tau) = \tilde{p}_1(\tau) - \tilde{p}_p(\tau) \ .$$

We have

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Lemma A.a.2:

a) $\lim\limits_{\tau\to\infty} \tilde{\psi}_h(\tau) = c_1 \frac{1}{2} \sqrt{\frac{b-1}{b+1}} - c_3 \frac{b-1}{b+1} + c_4 \frac{b-1}{\sqrt{b}(b+1)}$,

b) $\lim\limits_{\tau\to\infty} \dot{\tilde{\psi}}_h(\tau) = c_1 \frac{1}{2}$,

c) $\lim\limits_{t\to\infty} \tilde{n}_h(\tau) = c_1 \frac{b}{2\sqrt{b^2-1}} + c_3 \frac{1}{b+1} - c_4 \frac{\sqrt{b}}{b+1}$,

d) $\lim\limits_{\tau\to\infty} \tilde{p}_h(\tau) = -c_1 \frac{1}{2\sqrt{b^2-1}} + c_3 \frac{1}{b+1} + c_4 \frac{b+2}{\sqrt{b}\,(b+1)}$.

Proof: The Lemmas A.d.3, A.d.5, A.d.8, A.d.9, A.d.11 and A.d.12 immediately imply a)-d).

Lemma A.a.3:

a) $\left| \tilde{\psi}_h(2\tau_0) - \tilde{\psi}_h(\infty) \right| \le \text{const } \lambda \sqrt{\frac{b-1}{b+1}} \, (\ln\sqrt{b})^{3/2}$

b) $\left| \dot{\tilde{\psi}}_h(2\tau_0) - \dot{\tilde{\psi}}_h(\infty) \right| \le \text{const } \lambda (\ln\sqrt{b})^{3/2}$

c) $\left| \tilde{n}_h(2\tau_0) - \tilde{n}_h(\infty) \right| \le \text{const } \lambda \sqrt{\frac{b+1}{b-1}} \, (\ln\sqrt{b})^{3/2}$

d) $\left| \tilde{p}_h(2\tau_0) - \tilde{p}_h(\infty) \right| \le \text{const } \lambda \frac{1}{\sqrt{b^2-1}} \, (\ln\sqrt{b})^{3/2}$

Proof: The results can be shown by combining the Lemmas A.d.6, A.d.9 and A.d.11-17.

b) **Analysis of (A.18)**

The analysis of (A.18) is facilitated by the transformation

(A.b.1)
$$u = \mu + \nu ,$$
$$v = \mu - \nu .$$

Using (A.b.1) we obtain from (A.18):

(A.b.2)
$$\lambda\mu' = -\sqrt{2\bar{p}+1}\,\mu + \frac{1}{2}(f_4 + f_5),$$
$$\lambda\nu' = \sqrt{2\bar{p}+1}\,\nu + \frac{1}{2}(f_4 - f_5) .$$

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The behavior of solutions of (A.b.2) is characterized by

Lemma A.b.1: Let $\lambda y' = -\alpha(x)y + f(x)$, $x \in [a,b]$, $\alpha(x) > \underline{\alpha} > 0$. Then there is a solution y_h of the homogeneous equation and a particular solution y_p which satisfy

$$\|y_h\|_{L^\infty(a,b)} \leq 1, \quad y_h(a) = 1, \quad |y_h(b)| \leq \exp\left(-\frac{\underline{\alpha}}{\lambda}(b-a)\right),$$

$$\|y_p\|_{L^\infty(a,b)} \leq \frac{1}{\underline{\alpha}}\|f\|_{L^\infty(a,b)}, \quad \|y_p\|_{L^\infty(a,b)} \leq \lambda^{-1}\|f\|_{L^1(a,b)}.$$

Proof: We use $y_h(x) = \exp\left(-\lambda^{-1}\int_a^x \alpha(t)dt\right)$ and $y_p(x) = \lambda^{-1}\int_a^x \exp\left(-\lambda^{-1}\int_t^x \alpha(s)ds\right)f(t)dt$. An analogous result holds for $\alpha(x) \leq -\underline{\alpha} < 0$. Thus, we have

$$(A.b.3) \qquad\qquad u = c_5\mu_h + c_6\nu_h + \mu_p + \nu_p,$$

$$v = c_5\mu_h - c_6\nu_h + \mu_p - \nu_p,$$

where

$$(A.b.4) \qquad \|u\|_{L^\infty(x_0,1)}, \quad \|v\|_{L^\infty(x_0,1)} \leq \text{const}(|c_5| + |c_6| + \|f\|_{6,\lambda})$$

and

$$(A.b.5) \qquad \|u\|_{L^\infty(x_0,1)}, \quad \|v\|_{L^\infty(x_0,1)} \leq \text{const}(|c_5| + |c_6| + \|f\|_{6,\lambda}^*).$$

c) **The linearized reduced equations**

Integration of (A.19) gives

$$\bar{w} = c_7 \frac{1}{2\bar{p}+1} - c_9 \frac{x-x_0}{2(2\bar{p}+1)} + \bar{w}_p,$$

$$(A.c.1) \qquad\qquad \bar{t} = c_7 \frac{2}{2\bar{p}+1} + c_8 - c_9 \frac{x-x_0}{2\bar{p}+1} + \bar{t}_p,$$

$$\tilde{J} = c_9,$$

where

$$\bar{w}_p(x) = \frac{1}{2\bar{p}(x)+1}\int_1^x (2\bar{p}(t)+1)f_6(t)dt,$$

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$$\tilde{t}_p(x) = 2\tilde{w}_p(x) + \int_{x_0}^{x} (f_7(t) - 2f_6(t))dt$$

Since $(2\bar{p} + 1)$ is monotonically decreasing the estimates

$$\|\tilde{w}\|_{L^\infty(x_0,1)} < \text{const}(|c_7| + |c_9| + \|f\|_{6,\lambda}) ,$$

(A.c.2)
$$\|\tilde{t}\|_{L^\infty(x_0,1)} < \text{const}(|c_7| + |c_8| + |c_9| + \|f\|_{6,\lambda}) ,$$

$$|\tilde{J}| = |c_9|$$

hold.

d) Collection of technical results

We now state some useful general results.

Lemma A.d.1: Let y be the solution of the initial value problem

$$\dot{y} = a(\tau)y + f(\tau), \quad \tau > 0, \quad y(0) = y_0 ,$$

with $a, f \in C[0,\infty]^{*)}$, $a(\tau) < -\mu < 0$ for $\tau > 0$. Let $y_\infty = \lim_{y\to\infty} \frac{f(\tau)}{a(\tau)}$. Then

$$|y(\tau) - y_\infty| < |y_0 - y_\infty|e^{-\mu\tau} + \frac{1}{\mu} e^{-\mu\tau/2}\|f + ay_\infty\|_{L^\infty(0,\tau/2)} +$$

(A.d.1)
$$+ \frac{1}{\mu} \|f + ay_\infty\|_{L^\infty(\tau/2,\tau)}$$

holds.

Proof: We set $u = y - y_\infty$. Then u is the solution of

$$\dot{u} = a(\tau)u + h(\tau), \quad u(0) = y_0 - y_\infty ,$$

where $h(\tau) = f(\tau) + y_\infty a(\tau)$. Thus,

$$u = (y_0 - y_\infty)\exp(\int_0^\tau a(s)ds) + \int_0^\tau \exp(\int_s^\tau a(u)du)h(s)ds$$

We split the integral in the particular solution into two parts and use the upper bound of $a(\tau)$ to obtain the estimate

---

*) $g \in C[0,\infty]$ <==> ($g \in C[0,\infty)$ and $\lim_{\tau\to\infty} g(\tau)$ exists and is finite.)

$$|u(\tau)| \leq |y_0 - y_\infty|e^{-\mu\tau} + \int_0^\tau e^{-\mu(\tau-s)}|h(s)|ds \leq$$

$$\leq |y_0 - y_\infty|e^{-\mu\tau} + \|h\|_{L^\infty(0,\tau/2)}\frac{1}{\mu}(e^{-\mu\tau/2} - e^{-\mu\tau}) +$$

$$+ \|h\|_{L^\infty(\tau/2,\tau)}\frac{1}{\mu}(1 - e^{-\mu\tau/2}) ,$$

which implies (A.d.1).

Lemma A.d.2: Let y be the solution of

$$\dot{y} = a(\tau)y + f(\tau), \quad \tau > 0, \quad y(\infty) = y_\infty ,$$

with $a, f \in C[0,\infty]$, $a(\tau) \geq \mu > 0$ for $\tau > 0$ and $y_\infty = -\lim_{t\to\infty}\frac{f(\tau)}{a(\tau)}$. Then

(A.d.2)
$$|y(\tau) - y_\infty| \leq \frac{1}{\mu}\|f + ay_\infty\|_{L^\infty(\tau,\infty)} .$$

Proof: The function $u = y - y_\infty$ is the solution of

$$\dot{u} = a(\tau)u + h(\tau), \quad u(\infty) = 0 ,$$

where $h(\tau) = f(\tau) + y_\infty a(\tau)$. Thus

$$u = \int_\infty^\tau \exp(\int_s^\tau a(u)du)h(s)ds = -\int_\tau^\infty \exp(-\int_\tau^s a(u)du)h(s)ds$$

holds.

Using the lower bound of $a(\tau)$ we obtain the estimate

$$|u(\tau)| \leq \int_\tau^\infty e^{-\mu(s-\tau)}|h(s)|ds \leq \frac{1}{\mu}\|h\|_{L^\infty(\tau,\infty)} .$$

Lemma A.d.3: Let $g \in C[0,\infty)$; $f, h, f(\tau)\int_0^\tau g(s)ds \in C[0,\infty]$, and $f(\infty) = 0$. Then

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(A.d.3)
$$\lim_{\tau \to \infty} f(\tau) \int_0^\tau g(s)h(s)ds = \lim_{\tau \to \infty} h(\tau)f(\tau) \int_0^\tau g(s)ds$$

holds.

<u>Proof</u>: The limit on the right hand side of (A.d.3) exists. It remains to show

$$|y(\tau)| = \left| h(\tau)f(\tau) \int_0^\tau g(s)ds - f(\tau) \int_0^\tau g(s)h(s)ds \right| < \epsilon \quad \text{for} \quad \tau > \bar{\tau}(\epsilon) .$$

We set $\|h\|_{L^\infty(0,\infty)} = M_1$, $\|f(\tau) \int_0^\tau g(s)ds\|_{L^\infty(0,\infty)} = M_2$. Clearly

$$|y(\tau)| = \left| f(\tau) \int_0^\tau g(s)(h(\tau) - h(s))ds \right| \leq$$

$$\leq \left| f(\tau) \int_0^{\tilde{\tau}} g(s)(h(\tau) - h(s))ds \right| + \left| f(\tau) \int_{\tilde{\tau}}^\tau g(s)(h(\tau) - h(s)) \right|$$

holds. We choose $\tilde{\tau} = \tilde{\tau}(\epsilon)$ so that

$$|h(\tau) - h(s)| < \frac{\epsilon}{2M_2} \quad \text{for} \quad \tau,s > \tilde{\tau}$$

holds. Then we choose $\bar{\tau} = \bar{\tau}(\epsilon) > \tilde{\tau}$ such that

$$|f(\tau)| < \epsilon \left(4M_1 \int_0^{\tilde{\tau}} |g(s)|ds \right)^{-1} \quad \text{for} \quad \tau > \bar{\tau} .$$

This implies

$$|y(\tau)| < \epsilon \quad \text{for} \quad \tau > \bar{\tau} .$$

The following Lemmas are results on the behavior of the layer solutions.

<u>Lemma A.d.4</u>: There are positive constants $c_1 < c_2$ such that

$$c_1 \sqrt{\ln\sqrt{b}} < \dot{\psi}(0) < c_2 \sqrt{\ln\sqrt{b}}$$

holds.

Proof: The relations (2.11) imply

$$\dot{\hat{\psi}}(0) = \sqrt{2\left(\frac{2\sqrt{b}}{b-1} + \ln\sqrt{b} - \frac{b+1}{b-1}\right)} = \sqrt{2\left(\ln\sqrt{b} - \frac{\sqrt{b}-1}{\sqrt{b}+1}\right)} =$$

$$\sqrt{2\ln\sqrt{b}\, F(\sqrt{b})}, \quad \text{with} \quad F(\xi) = 1 - \frac{\xi-1}{(\xi+1)\ln\xi}$$

Elementary calculations give

$$F \in C[1,\infty], \quad F(\xi) > c > 0 \quad \text{for} \quad \xi \in [1,\infty]$$

This implies the existence of $c_1, c_2 > 0$ such that

$$\frac{c_1^2}{2} < F(\sqrt{b}) < \frac{c_2^2}{2}\,.$$

Lemma A.d.5: $\sqrt{\frac{b-1}{b+1}} < \left\|\frac{\dot{\hat{\psi}}}{\hat{n}-\hat{p}}\right\|_{L^\infty[0,\infty)} < \dot{\hat{\psi}}(0), \quad \lim_{\tau\to\infty}\frac{\dot{\hat{\psi}}}{\hat{n}-\hat{p}} = -\sqrt{\frac{b-1}{b+1}}$

Proof: Let $H(\hat{\psi}) = \left(\frac{\dot{\hat{\psi}}}{\hat{n}-\hat{p}}\right)^2$. With (2.11) we get

$$H(-\ln\sqrt{b}) = \dot{\hat{\psi}}(0)^2$$

To verify the second assertion of the lemma we use the rule of de l'Hopital and the relations

(A.d.4)
$$\frac{d\hat{\psi}}{d\hat{\psi}} = \frac{\hat{n}-\hat{p}}{\dot{\hat{\psi}}}, \quad \frac{d\hat{n}}{d\hat{\psi}} = \frac{b}{b-1} + \hat{n}, \quad \frac{d\hat{p}}{d\hat{\psi}} = -\frac{1}{b-1} - \hat{p}\,,$$

which follow from (2.11):

$$\lim_{\hat{\psi}\to 0} H(\hat{\psi}) = \lim_{\hat{\psi}\to 0}\frac{\dot{\hat{\psi}}^2}{(\hat{n}-\hat{p})^2} = \lim_{\hat{\psi}\to 0}\frac{2(\hat{n}-\hat{p})}{2(\hat{n}-\hat{p})\left(\frac{b+1}{b-1} + \hat{n} + \hat{p}\right)} = \frac{b-1}{b+1}$$

To complete the proof of Lemma A.d.5 we show that $H(\hat{\psi})$ is monotonically decreasing:

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$$\frac{dH}{d\hat{\psi}} = \frac{2M(\hat{\psi})}{(\hat{n} - \hat{p})^3}, \quad \text{with} \quad M(\hat{\psi}) = (\hat{n} - \hat{p})^2 - \dot{\hat{\psi}}^2\left(\frac{b+1}{b-1} + \hat{n} + \hat{p}\right)$$

Since $\hat{n} - \hat{p} < 0$ we have to show that $M(\hat{\psi}) > 0$. (A.d.4) gives $\frac{dM}{d\hat{\psi}} = -\dot{\hat{\psi}}^2(\hat{n} - \hat{p} + 1) < 0$, for $\hat{\psi} \in [-\ell n \sqrt{b}, 0]$, which implies that M is monotonically decreasing on $[-\ell n \sqrt{b}, 0]$ and, thus, $M(\hat{\psi}) > 0$ there, because $M(0) = 0$ holds.

<u>Lemma A.d.6:</u> Let $\tau_0 = \dot{\hat{\psi}}(0) \ell n \frac{1}{\lambda}$. Then

$$|\dot{\hat{\psi}}|_{L^\infty(i\tau_0, \infty)} < \text{const } \lambda^i \dot{\hat{\psi}}(0)^2, \quad |\dot{\hat{\psi}}|_{L^\infty(i\tau_0, \infty)} < \lambda^i \dot{\hat{\psi}}(0),$$

$$|\hat{n}|_{L^\infty(i\tau_0, \infty)} < \text{const } \lambda^i \frac{b}{b-1} \dot{\hat{\psi}}(0)^2, \quad |\hat{p}|_{L^\infty(i\tau_0, \infty)} < \text{const } \lambda^i \frac{1}{b-1} \dot{\hat{\psi}}(0)^2$$

hold for $i > 0$.

<u>Proof:</u> $z = \dot{\hat{\psi}}$ is the solution of

$$\dot{z} = \frac{\hat{n} - \hat{p}}{\dot{\hat{\psi}}} z, \quad z(0) = \dot{\hat{\psi}}(0)$$

An application of Lemma A.d.1 with $\mu = \dot{\hat{\psi}}(0)^{-1}$ (see Lemma A.d.5) yields

(A.d.5) $$\dot{\hat{\psi}}(\tau) < \dot{\hat{\psi}}(0) \exp(-\tau/\dot{\hat{\psi}}(0)).$$

Taylor's theorem, Lemma A.d.5 and (A.d.4) imply

$$|\hat{\psi}(\tau)| < \dot{\hat{\psi}}(0)\dot{\hat{\psi}}(\tau)$$

$$|\hat{n}(\tau)| < \text{const } \frac{b}{b-1} \dot{\hat{\psi}}(0)\dot{\hat{\psi}}(\tau)$$

$$|\hat{p}(\tau)| < \text{const } \frac{1}{b-1} \dot{\hat{\psi}}(0)\dot{\hat{\psi}}(\tau)$$

This completes the proof of Lemma A.d.6.

Lemma A.d.7: $\left| \dfrac{\hat{n} - \hat{p}}{\dot{\hat{\psi}}} + \sqrt{\dfrac{b+1}{b-1}} \right|_{L^\infty(i\tau_0, \infty)} < \text{const } \lambda^i \dot{\hat{\psi}}(0)^2$ holds for $i > 0$.

Proof: We set $H(\hat{\psi}) = \left(\dfrac{\hat{n} - \hat{p}}{\dot{\hat{\psi}}} \right)^2 - \dfrac{(F'(\hat{\psi}))^2}{2F(\hat{\psi})}$, where $F(\hat{\psi}) = \dfrac{b}{b-1} e^{\hat{\psi}} + \dfrac{1}{b-1} e^{-\hat{\psi}} - \hat{\psi} - \dfrac{b+1}{b-1}$.

An application of Taylor's theorem yields

$$(F'(\hat{\psi}))^2 = \hat{\psi}^2 \left(\dfrac{b+1}{b-1} \right)^2 \left[1 + 0\left(\hat{\psi} \dfrac{b-1}{b+1} \right) \right] ,$$

$$2F(\hat{\psi}) = \hat{\psi}^2 \dfrac{b+1}{b-1} \left[1 + 0\left(\hat{\psi} \dfrac{b-1}{b+1} \right) \right] ,$$

which implies

$$H(\hat{\psi}) = \dfrac{b+1}{b-1} \left[1 + 0\left(\hat{\psi} \dfrac{b-1}{b+1} \right) \right] .$$

Hence

$$\dfrac{\hat{n} - \hat{p}}{\dot{\hat{\psi}}} = - \sqrt{\dfrac{b+1}{b-1}} \left[1 + 0\left(\hat{\psi} \dfrac{b-1}{b+1} \right) \right] = - \sqrt{\dfrac{b+1}{b-1}} + 0(\hat{\psi}) .$$

The assertion follows from Lemma A.d.6.

Lemma A.d.8: Let $y = \dot{\hat{\psi}}(\tau) \displaystyle\int_0^\tau \dot{\hat{\psi}}(s)^{-1} ds$. Then

a) $\displaystyle\lim_{\tau \to \infty} y(\tau) = \sqrt{\dfrac{b-1}{b+1}}$

b) $|y|_{L^\infty(0,\infty)} < \dot{\hat{\psi}}(0)$

Proof: a) de l'Hopital's theorem and Lemma A.d.5 give:

$$\lim_{\tau \to \infty} y(\tau) = \lim_{\tau \to \infty} \left(- \dfrac{\hat{n} - \hat{p}}{\dot{\hat{\psi}}^2} \right)^{-1} \dfrac{1}{\dot{\hat{\psi}}} = - \lim_{\tau \to \infty} \dfrac{\dot{\hat{\psi}}}{\hat{n} - \hat{p}} = \sqrt{\dfrac{b-1}{b+1}} .$$

b) y is the solution of

$$\dot{y} = \dfrac{\hat{n} - \hat{p}}{\dot{\hat{\psi}}} y + 1, \quad y(0) = 0 .$$

Since $\dot{y} = 0$ holds at a relative minimum or maximum of y in $(0, \infty)$ we obtain

$$|y| = \left|\frac{\dot{\hat{\psi}}}{\mathrm{x} \underbrace{\quad}_{n-p} \mathrm{x}}\right| < \dot{\hat{\psi}}(0)$$

From $y(0) = 0$ and a) we conclude b).

Lemma A.d.9: Let $y = \dot{\hat{\psi}}^2(\tau) \int_0^\tau \dot{\hat{\psi}}^{-2}(s)ds$. Then

a) $\lim\limits_{\tau \to \infty} y(\tau) = \frac{1}{2}\sqrt{\frac{b-1}{b+1}}$

b) $\|y\|_{L^\infty(0,\infty)} < \dot{\hat{\psi}}(0)$

c) $\left\|y - \frac{1}{2}\sqrt{\frac{b-1}{b+1}}\right\|_{L^\infty(2i\tau_0,\infty)} < \text{const } \lambda^i \sqrt{\frac{b-1}{b+1}} \, \dot{\hat{\psi}}^3(0)$ for $i > 0$.

Proof: a) From de l'Hopital's theorem and Lemma A.d.5 we derive:

$$\lim_{\tau \to \infty} y(\tau) = \lim_{\tau \to \infty} \left(-2\frac{\hat{n}-\hat{p}}{\dot{\hat{\psi}}^3}\right)^{-1} \frac{1}{\dot{\hat{\psi}}^2} = \frac{1}{2}\sqrt{\frac{b-1}{b+1}}.$$

b) $\dot{\hat{\psi}}$ is positive and monotonically decreasing. Thus, Lemma A.d.8 gives

$$y(\tau) < \dot{\hat{\psi}}(\tau) \int_0^\tau \dot{\hat{\psi}}^{-1}(s)ds < \dot{\hat{\psi}}(0).$$

c) y is the solution of

$$\dot{y} = 2\frac{\hat{n}-\hat{p}}{\dot{\psi}} y + 1, \quad y(0) = 0.$$

An application of Lemma A.d.1 with $\mu = \dfrac{2}{\dot{\psi}(0)}$ (see Lemma A.d.5) gives

$$\left|y(\tau) - \frac{1}{2}\sqrt{\frac{b-1}{b+1}}\right| < \frac{1}{2}\sqrt{\frac{b-1}{b+1}}\exp\left(-\frac{2\tau}{\overset{\cdot}{\psi}(0)}\right) +$$

$$+ \frac{1}{2}\overset{\cdot}{\hat{\psi}}(0)\exp\left(-\frac{\tau}{\overset{\cdot}{\psi}(0)}\right)\left|1 + \frac{\hat{n}-\hat{p}}{\overset{\cdot}{\psi}(0)}\sqrt{\frac{b-1}{b+1}}\right|_{L^{\infty}(0,\infty)} +$$

$$+ \frac{1}{2}\overset{\cdot}{\hat{\psi}}(0)\left|1 + \frac{\hat{n}-\hat{p}}{\overset{\cdot}{\psi}}\sqrt{\frac{b-1}{b+1}}\right|_{L^{\infty}(\tau/2,\infty)} \quad,$$

which implies

$$\left|y - \frac{1}{2}\sqrt{\frac{b-1}{b+1}}\right|_{L^{\infty}(2i\tau_0,\infty)} < \text{const}\left[\sqrt{\frac{b-1}{b+1}}\lambda^{4i} + \overset{\cdot}{\hat{\psi}}(0)\lambda^{2i} + \overset{\cdot}{\psi}(0)\sqrt{\frac{b-1}{b+1}}\overset{\cdot}{\psi}^2(0)\lambda^{i}\right] \quad.$$

<u>Lemma A.d.10.</u> Let $y(\tau) = \dfrac{1}{\overset{\cdot}{\psi}(\tau)}\displaystyle\int_{\tau}^{\infty}\overset{\cdot}{\hat{\psi}}(s)\,ds.$ Then

$$\left|y\right|_{L^{\infty}(0,\infty)} < \text{const}\sqrt{\ell n\sqrt{b}}$$

<u>Proof:</u> Obviously $y(\tau) = -\overset{\wedge}{\psi}(\tau)/\overset{\cdot}{\psi}(\tau)$ and y is the solution of

$$\overset{\cdot}{y} = -\frac{\hat{n}-\hat{p}}{\overset{\cdot}{\psi}}y - 1, \quad y(0) = -\overset{\wedge}{\psi}(0)/\overset{\cdot}{\psi}(0)$$

y assumes its maximum at $\tau = 0$ or at a stationary point. Using Lemma A.d.4 we obtain

$$y(0) = \frac{\ell n\sqrt{b}}{\overset{\cdot}{\psi}(0)} < \text{const}\sqrt{\ell n\sqrt{b}} \quad.$$

For stationary points

$$\left|y(\tau)\right| = \left|\frac{\overset{\cdot}{\overset{\wedge}{\psi}}}{\hat{n}-\hat{p}}\right| < \overset{\cdot}{\hat{\psi}}(0) < \text{const}\sqrt{\ell n\sqrt{b}}$$

holds by Lemma A.d.5.

<u>Lemma A.d.11</u>: Let $y = \dfrac{1 - e^{\hat{\psi}}}{\dfrac{\hat{\psi}}{}}$. Then

a) $\lim\limits_{\tau \to \infty} y(\tau) = \sqrt{\dfrac{b-1}{b+1}}$,

b) $\|y\|_{L^\infty(0,\infty)} < \text{const } \sqrt{\ell n \sqrt{b}}$,

c) $\|y - \sqrt{\dfrac{b-1}{b+1}}\|_{L^\infty(i\tau_0,\infty)} < \text{const } \lambda^{i}\dot{\hat{\psi}}^3(0)$ for $i > 0$.

<u>Proof</u>: a) We apply de l'Hopital's theorem and Lemma A.d.5:

$$\lim\limits_{\tau \to \infty} y(\tau) = \lim\limits_{\psi \to 0} \frac{1 - e^{\hat{\psi}}}{\hat{\psi}} = - \lim\limits_{\psi \to 0} e^{\hat{\psi}} \frac{\dot{\hat{\psi}}}{\hat{n} - \hat{p}} = \sqrt{\frac{b-1}{b+1}}$$

b) y is the solution of

$$\dot{y} = - \frac{\hat{n} - \hat{p}}{\hat{\psi}} y - e^{\hat{\psi}}, \quad y(\infty) = \sqrt{\frac{b-1}{b+1}}$$

As in the proof of Lemma A.d.10 we have to estimate $|y(\tau)|$ at $\tau = 0$ and at stationary points:

$$y(0) = \left(1 - \frac{1}{\sqrt{b}}\right)\dot{\hat{\psi}}^{-1}(0) < \text{const } \frac{\sqrt{b} - 1}{\sqrt{b}\sqrt{\ell n \sqrt{b}}}$$

Similarly to the proof of Lemma A.d.4 we obtain

$$y(0) < \text{const } \sqrt{\ell n \sqrt{b}} .$$

For stationary points we have

$$|y| = e^{\hat{\psi}}\left|\frac{\dot{\hat{\psi}}}{n - p}\right| < \dot{\hat{\psi}}(0) < \text{const } \sqrt{\ell n \sqrt{b}} .$$

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c) We apply Lemma A.d.2 with $\mu = \dot{\hat{\psi}}^{-1}(0)$ and obtain

$$\left| y(\tau) - \sqrt{\frac{b-1}{b+1}} \right| < \dot{\hat{\psi}}(0) \left\| \sqrt{\frac{b-1}{b+1}} \frac{\hat{p} - \hat{n}}{\dot{\hat{\psi}}} - e^{\hat{\psi}} \right\|_{L^\infty(\tau,\infty)} \quad ,$$

which implies

$$\left\| y - \sqrt{\frac{b-1}{b+1}} \right\|_{L^\infty(i\tau_0,\infty)} < \dot{\hat{\psi}}(0) \sqrt{\frac{b-1}{b+1}} \left\| \frac{\hat{n} - \hat{p}}{\dot{\hat{\psi}}} \right\| + \sqrt{\frac{b+1}{b-1}} \left\| \quad \right\|_{L^\infty(i\tau_0,\infty)} + \dot{\hat{\psi}}(0) \left\| 1 - e^{\hat{\psi}} \right\|_{L^\infty(i\tau_0,\infty)} <$$

$$< const\ \dot{\hat{\psi}}(0) \left[\sqrt{\frac{b-1}{b+1}} \lambda^i \dot{\hat{\psi}}^2(0) + \lambda^i \dot{\hat{\psi}}^2(0) \right]$$

by the Lemmas A.d.6 and A.d.7.

Lemma A.d.12: Let $y = \dfrac{e^{-\hat{\psi}} - 1}{\sqrt{b}\ \dot{\hat{\psi}}}$. Then

a) $\displaystyle \lim_{\tau \to \infty} y(\tau) = \sqrt{\frac{b-1}{b(b+1)}}$

b) $\|y\|_{L^\infty(0,\infty)} < const \sqrt{\ell n \sqrt{b}}$

c) $\left\| y - \sqrt{\dfrac{b-1}{b(b+1)}} \right\|_{L^\infty(i\tau_0,\infty)} < const\ \lambda^i \dot{\hat{\psi}}^3(0)$ \quad for $i > 0$.

Proof: a) $y(\tau) = \dfrac{e^{-\hat{\psi}}}{\sqrt{b}} \cdot \dfrac{1 - e^{\hat{\psi}}}{\dot{\hat{\psi}}}$. Lemma A.d.11a) yields the result.

b) follows from Lemma A.d.11b) and from $e^{-\hat{\psi}} < \sqrt{b}$.

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c) $\left\| y - \sqrt{\dfrac{b-1}{b(b+1)}} \right\|_{L^\infty(i\tau_0,\infty)} < \left\| \dfrac{e^{-\hat\psi}}{\sqrt{b}} \cdot \dfrac{1 - e^{\hat\psi}}{\dot{\hat\psi}} - e^{-\hat\psi} \sqrt{\dfrac{b-1}{b(b+1)}} \right\|_{L^\infty(i\tau_0,\infty)} +$

$+ \left\| e^{-\hat\psi} \sqrt{\dfrac{b-1}{b(b+1)}} - \sqrt{\dfrac{b-1}{b(b+1)}} \right\|_{L^\infty(i\tau_0,\infty)} < \left\| \dfrac{1 - e^{\hat\psi}}{\dot{\hat\psi}} - \sqrt{\dfrac{b-1}{b+1}} \right\|_{L^\infty(i\tau_0,\infty)} +$

$+ \sqrt{\dfrac{b-1}{b(b+1)}} \left\| e^{-\hat\psi} - 1 \right\|_{L^\infty(i\tau_0,\infty)} < \text{const } \lambda^i \left(\dot{\hat\psi}^3(0) + \sqrt{\dfrac{b-1}{b(b+1)}}\, \dot{\hat\psi}^2(0) \right).$

Lemma A.d.13: Let $y(\tau) = \dot{\hat\psi}(\tau) \displaystyle\int_0^\tau \dfrac{1 - e^{\hat\psi}(s)}{\dot{\hat\psi}^2(s)}\, ds$. Then

a) $\displaystyle\lim_{\tau\to\infty} y(\tau) = \dfrac{b-1}{b+1}$

b) $\left\| y - \dfrac{b-1}{b+1} \right\|_{L^\infty(2i\tau_0,\infty)} < \text{const } \lambda^i \dot{\hat\psi}^4(0)$ for $i > 0$.

Proof: The Lemmas A.d.3, A.d.8 and A.d.11 imply a). y is the solution of

$$\dot{y} = \dfrac{\hat{n} - \hat{p}}{\dot{\hat\psi}}\, y + \dfrac{1 - e^{\hat\psi}}{\dot{\hat\psi}}, \quad y(0) = 0$$

An application of Lemma A.d.1 with $\mu = \dot{\hat\psi}^{-1}(0)$ (see Lemma A.d.5) yields

$$\left| y(\tau) - \dfrac{b-1}{b+1} \right| < \dfrac{b-1}{b+1} \exp(-\dot{\hat\psi}^{-1}(0)\tau) + \dot{\hat\psi}(0)\exp\left(-\dot{\hat\psi}^{-1}(0)\,\dfrac{\tau}{2}\right) \left\| \dfrac{1 - e^{\hat\psi}}{\dot{\hat\psi}} + \dfrac{\hat{n} - \hat{p}}{\dot{\hat\psi}}\, \dfrac{b-1}{b+1} \right\|_{L^\infty(0,\infty)}$$

$$+ \dot{\hat\psi}(0) \left\| \dfrac{1 - e^{\hat\psi}}{\dot{\hat\psi}} + \dfrac{\hat{n} - \hat{p}}{\dot{\hat\psi}}\, \dfrac{b-1}{b+1} \right\|_{L^\infty(\tau/2,\infty)}.$$

Thus,

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$$\|y - \frac{b-1}{b+1}\|_{L^\infty(2i\tau_0, \infty)} < \lambda^{2i} \frac{b-1}{b+1} + \dot{\hat{\psi}}(0)\lambda^i \text{const} \sqrt{\ln\sqrt{b}} +$$

$$+ \dot{\hat{\psi}}(0)\|\frac{1-e^{\hat{\psi}}}{\hat{\psi}^2} - \sqrt{\frac{b-1}{b+1}}\|_{L^\infty(i\tau_0, \infty)} + \dot{\hat{\psi}}(0)\frac{b-1}{b+1}\|\sqrt{\frac{b+1}{b-1}} + \frac{\hat{n}-\hat{p}}{\hat{\psi}^2}\|_{L^\infty(i\tau_0, \infty)} <$$

$$< \text{const } \lambda^i[\lambda^i \frac{b-1}{b+1} + \dot{\hat{\psi}}(0)\sqrt{\ln\sqrt{b}} + \dot{\hat{\psi}}^4(0) + \frac{b-1}{b+1}\dot{\hat{\psi}}^3(0)] .$$

<u>Lemma A.d.14</u>: Let $y(\tau) = \dot{\hat{\psi}}(\tau) \int_0^\tau \dfrac{e^{-\hat{\psi}(s)} - 1}{\sqrt{b}\,\hat{\psi}^2(s)} ds$. Then

a) $\lim\limits_{\tau \to \infty} y(\tau) = \dfrac{b-1}{\sqrt{b}(b+1)}$,

b) $\|y - \dfrac{b-1}{\sqrt{b}(b+1)}\|_{L^\infty(2i\tau_0, \infty)} < \text{const } \lambda^i\dot{\hat{\psi}}^4(0)$ for $i > 0$.

The <u>proof</u> of this Lemma proceeds analogously to that of Lemma A.d.13 and is therefore omitted.

<u>Lemma A.d.15</u>: Let $y(\tau) = \dot{\hat{\psi}}(\tau)(\hat{n}(\tau) - \hat{p}(\tau)) \int_0^\tau \dot{\hat{\psi}}^{-2}(s) ds$. Then

a) $\lim\limits_{\tau \to \infty} y(\tau) = -\dfrac{1}{2}$

b) $\|y + \dfrac{1}{2}\|_{L^\infty(2i\tau_0, \infty)} < \text{const } \lambda^i\dot{\hat{\psi}}^3(0)$ for $i > 0$.

<u>Proof</u>: a) $y = \dfrac{\hat{n}-\hat{p}}{\hat{\psi}^2} \dot{\hat{\psi}}^2 \int_0^\tau \dot{\hat{\psi}}^{-2}(s) ds$. We combine the Lemmas A.d.5 and A.d.9a) to obtain a).

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b) $\|y + \frac{1}{2}\|_{L^\infty(2i\tau_0,\infty)} < \|y - \frac{\hat{n} - \hat{p}}{\overset{\hat{}}{\underset{\psi}{i}}} \frac{1}{2} \sqrt{\frac{b-1}{b+1}}\|_{L^\infty(2i\tau_0,\infty)} +$

$$+ \|\frac{\hat{n} - \hat{p}}{\overset{\hat{}}{\underset{\psi}{i}}} \frac{1}{2} \sqrt{\frac{b-1}{b+1}} + \frac{1}{2}\|_{L^\infty(2i\tau_0,\infty)} <$$

$$< \|\frac{\hat{n} - \hat{p}}{\overset{\hat{}}{\underset{\psi}{i}}}\|_{L^\infty(2i\tau_0,\infty)} \overset{\cdot\hat{}}{\psi}{}^2(\tau) \int_0^\tau \overset{\cdot\hat{}}{\psi}{}^{-2}(s)\,ds - \frac{1}{2}\sqrt{\frac{b-1}{b+1}}\|_{L^\infty(2i\tau_0,\infty)} +$$

$$+ \frac{1}{2}\sqrt{\frac{b-1}{b+1}}\|\frac{\hat{n} - \hat{p}}{\overset{\hat{}}{\underset{\psi}{i}}} + \sqrt{\frac{b+1}{b-1}}\|_{L^\infty(2i\tau_0,\infty)}$$

$$< \text{const } \lambda^i \left(\sqrt{\frac{b+1}{b-1}} \cdot \sqrt{\frac{b-1}{b+1}} \overset{\cdot\hat{}}{\psi}{}^3(0) + \lambda^i \sqrt{\frac{b-1}{b+1}} \overset{\cdot\hat{}}{\psi}{}^2(0) \right)$$

Lemma A.d.16: Let $y(\tau) = (\hat{n}(\tau) - \hat{p}(\tau)) \int_0^\tau \frac{1 - e^{\hat{\psi}(s)}}{\overset{\hat{}}{\underset{\psi}{i}}{}^2(s)}\,ds.$ Then

a) $\lim_{\tau \to \infty} y(\tau) = -\sqrt{\frac{b-1}{b+1}}$

b) $\|y + \sqrt{\frac{b-1}{b+1}}\|_{L^\infty(2i\tau_0,\infty)} < \text{const } \lambda^i \sqrt{\frac{b+1}{b-1}} \overset{\cdot\hat{}}{\psi}{}^4(0)$ for $i > 0.$

Lemma A.d.17: Let $y(\tau) = (\hat{n}(\tau) - \hat{p}(\tau)) \int_0^\tau \frac{e^{-\hat{\psi}(s)} - 1}{\sqrt{b} \cdot \overset{\hat{}}{\underset{\psi}{i}}(s)}\,ds.$ Then

a) $\lim_{\tau \to \infty} y(\tau) = -\sqrt{\frac{b-1}{b(b+1)}},$

(b) $\|y + \sqrt{\frac{b-1}{b(b+1)}}\|_{L^\infty(2i\tau_0,\infty)} < \text{const } \lambda^i \sqrt{\frac{b+1}{b-1}} \overset{\cdot\hat{}}{\psi}{}^4(0)$ for $i > 0.$

The proofs of the Lemmas A.d.16, A.d.17 are similar to that of Lemma A.d.15.

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<u>Lemma A.d.18</u>: a) $\left|\dot{\hat{\psi}}_1\right|_{L^\infty(0,\infty)} < \text{const}\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}(\ell n\sqrt{b})^{5/2}\right)$

b) $\left|\dot{\hat{\psi}}_1\right|_{L^\infty(0,\infty)} < \text{const}\left(\frac{1}{b} + \delta^4\sqrt{b}(\ell n\sqrt{b})^{5/2}\right)$,

c) $\left|\hat{n}_1\right|_{L^\infty(0,\infty)} < \text{const}\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}(\ell n\sqrt{b})^{5/2}\right)$,

d) $\left|\hat{P}_1\right|_{L^\infty(0,\infty)} < \text{const}\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}(\ell n\sqrt{b})^{3/2}\right)$.

<u>Proof</u>: The estimates

(A.d.6)
$$\left|\hat{n}\right|_{L^1(0,\infty)},\ \left|\hat{p}\right|_{L^1(0,\infty)} < \text{const}\sqrt{\ell n\sqrt{b}}\ ,$$

(A.d.7)
$$\left|\tau\dot{\hat{\psi}}\right|_{L^1(0,\infty)} < \text{const}(\ell n\sqrt{b})^{3/2}.$$

hold. Since

$$\left|\hat{n}\right|_{L^1(0,\infty)},\ \left|\hat{p}\right|_{L^1(0,\infty)} < \left|\hat{n} - \hat{p}\right|_{L^1(0,\infty)} = \dot{\hat{\psi}}(0)$$

holds, (A.d.6) follows. Using (A.d.5) in the proof of Lemma A.d.6 we get

$$\left|\tau\dot{\hat{\psi}}\right|_{L^1(0,\infty)} < \dot{\hat{\psi}}(0)\left|\tau\ \exp(-\tau/\dot{\hat{\psi}}(0))\right|_{L^1(0,\infty)} = \dot{\hat{\psi}}^3(0)$$

which implies (A.d.7). Now we are able to find estimates on the functions in (2.16).
Using $\bar{p}'(0) = -\left(\frac{b}{(b-1)^2} - \delta^4\right)\frac{b-1}{b+1}$ and the monotonicity and sign of $\dot{\hat{\psi}}$ we get

$$\left|\hat{n}_p\right|_{L^\infty(0,\infty)} < \text{const}\left(\frac{b}{(b-1)^2} + \delta^4\right)\frac{b-1}{b+1}\left(\sqrt{\ell n\sqrt{b}} + (\ell n\sqrt{b})^{3/2}\right) <$$

$$< \text{const}\left(\frac{1}{\sqrt{b-1}} + \delta^4(\ell n\sqrt{b})^{3/2}\right)\ ,$$

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$$\|\hat{p}_p\|_{L^\infty(0,\infty)} \leq \mathrm{const}\left(\frac{b}{(b-1)^2} + \delta^4\right)\frac{b-1}{b+1}\sqrt{b}\ (\sqrt{\ln\sqrt{b}} + (\ln\sqrt{b})^{3/2}) \leq$$

$$\leq \mathrm{const}\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{3/2}\right)\ .$$

We set $H(\tau) = \bar{p}_1(0)(e^{\hat{\psi}(\tau)} - e^{-\hat{\psi}(\tau)}) + \hat{n}_p(\tau) - \hat{p}_p(\tau)$, use

$$\bar{p}_1(0) = -\left(\frac{b}{(b-1)^2} - \delta^4\right)\left(\frac{b-1}{b+1}\right)^2\dot{\hat{\psi}}(0)$$

and obtain

$$\|H\|_{L^\infty(0,\infty)} \leq \mathrm{const}\left[\sqrt{b}\ \left(\frac{b}{(b-1)^2} + \delta^4\right)\left(\frac{b-1}{b+1}\right)^2\sqrt{\ln\sqrt{b}} + \frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}(\ln\sqrt{b})^{3/2}\right] \leq$$

$$\leq \mathrm{const}\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{3/2}\right)\ .$$

This estimate and the Lemmas A.d.8 and A.d.10 imply

$$\|\hat{\psi}_p\|_{L^\infty(0,\infty)} \leq \mathrm{const}\ \ln\sqrt{b}\ \left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{3/2}\right) \leq$$

$$\leq \mathrm{const}\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right)$$

Obviously

$$\dot{\hat{\psi}}_p = \frac{\hat{n} - \hat{p}}{\dot{\psi}}\hat{\psi}_p - \dot{\hat{\psi}}^{-1}(\tau)\int_\tau^\infty \dot{\hat{\psi}}(s)H(s)ds\ ,$$

which implies

$$\|\dot{\hat{\psi}}_p\|_{L^\infty(0,\infty)} \leq \mathrm{const}\left[\sqrt{\frac{b+1}{b-1}}\ \left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right) + \right.$$

$$\left. + \sqrt{\ln\sqrt{b}}\ \left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{3/2}\right)\right] \leq$$

$$\leq \mathrm{const}\left(\frac{1}{b} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right)$$

The constant c in (2.15) satisfies (see Lemma 2.1)

$$|c| = |2(\frac{b}{(b-1)^2} - \delta^4)(\frac{b-1}{b+1})^2\dot{\psi}(0)| < const(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ell n\sqrt{b}})$$

Thus, we have

$$|\hat{\psi}_1|_{L^\infty(0,\infty)} < const(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ell n\sqrt{b}} + \frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ell n\sqrt{b})^{5/2}) <$$

$$< const(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ell n\sqrt{b})^{5/2})$$

$$|\dot{\hat{\psi}}_1|_{L^\infty(0,\infty)} = |c\frac{\hat{n} - \hat{p}}{\ell} + \dot{\hat{\psi}}_p|_{L^\infty(0,\infty)} < const((\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ell n\sqrt{b}})\frac{1}{\sqrt{\ell n\sqrt{b}}} +$$

$$+ \frac{1}{b} + \delta^4\sqrt{b}(\ell n\sqrt{b})^{5/2}) < const(\frac{1}{b} + \delta^4\sqrt{b}\ (\ell n\sqrt{b})^{5/2}) ,$$

$$|\hat{n}_1|_{L^\infty(0,\infty)} < const[\frac{b}{b-1}\ (\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}(\ell n\sqrt{b})^{5/2})$$

$$+ (\frac{b}{(b-1)^2} + \delta^4)(\frac{b-1}{b+1})^2\sqrt{\ell n\sqrt{b}} + \frac{1}{\sqrt{b-1}} + \delta^4(\ell n\sqrt{b})^{3/2}] <$$

$$< const(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\ (\ell n\sqrt{b})^{5/2}) ,$$

$$|\hat{p}_1|_{L^\infty(0,\infty)} < const[\frac{\sqrt{b}}{b-1}\ (\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ell n\sqrt{b})^{5/2})$$

$$+ \sqrt{b}\ (\frac{b}{(b-1)^2} + \delta^4)(\frac{b-1}{b+1})^2\sqrt{\ell n\sqrt{b}} + \frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\ (\ell n\sqrt{b})^{3/2}] <$$

$$< const(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}\ (\ell n\sqrt{b})^{3/2}) .$$

<u>Lemma A.d.19</u>: a) $|\hat{\dot{\psi}}_1|_{L^\infty(2i\tau_0,\infty)} < \text{const } \lambda^i\left(\ln\frac{1}{\lambda}\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\,(\ln\sqrt{b})^{5/2}\right)$

b) $|\hat{\dot{\psi}}_1|_{L^\infty(2i\tau_0,\infty)} < \text{const } \lambda^i\left(\frac{1}{b}\ln\frac{1}{\lambda} + \delta^4\sqrt{b}(\ln\sqrt{b})^{5/2}\right)$,

c) $|\hat{\dot{n}}_1|_{L^\infty(2i\tau_0,\infty)} < \text{const } \lambda^i\left(\frac{1}{\sqrt{b-1}}\ln\frac{1}{\lambda} + \delta^4\sqrt{b}(\ln\sqrt{b})^{5/2}\right)$,

d) $|\hat{p}_1|_{L^\infty(2i\tau_0,\infty)} < \text{const } \lambda^i\frac{1}{\sqrt{b-1}}\ln\frac{1}{\lambda}$.

hold for $i > 0$.

<u>Proof</u>: The estimates

(A.d.8) $|\hat{n}|_{L^1(i\tau_0,\infty)}$, $|\hat{p}|_{L^1(i\tau_0,\infty)} < \text{const } \lambda^i\sqrt{\ln\sqrt{b}}$

(A.d.9) $|\tau\hat{\psi}|_{L^1(i\tau_0,\infty)} < \text{const } \lambda^i\ln\frac{1}{\lambda}\,(\ln\sqrt{b})^{3/2}$

hold. The proofs of (A.d.8) and (A.d.9) are similar to those of (A.d.6) and (A.d.7) and therefore omitted. Proceeding along the lines of the proof of Lemma A.d.18 gives

$|\hat{n}_p|_{L^\infty(i\tau_0,\infty)} < \text{const } \lambda^i\ln\frac{1}{\lambda}\left(\frac{1}{\sqrt{b-1}} + \delta^4(\ln\sqrt{b})^{3/2}\right)$,

$|\hat{p}_p|_{L^\infty(i\tau_0,\infty)} < \text{const } \lambda^i\ln\frac{1}{\lambda}\left(\frac{1}{\sqrt{b-1}} + \delta^4(\ln\sqrt{b})^{3/2}\right)$,

$|H|_{L^\infty(i\tau_0,\infty)} < \text{const } \lambda^i\ln\frac{1}{\lambda}\left(\frac{1}{\sqrt{b-1}} + \delta^4(\ln\sqrt{b})^{3/2}\right)$.

Now we apply Lemma A.d.10 and obtain

(A.d.10) $|\hat{\psi}_p|_{L^\infty(2i\tau_0,\infty)} < \text{const }\sqrt{\ln\sqrt{b}}\,|y|_{L^\infty(2i\tau_0,\infty)}$,

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where $y(\tau) = \dot{\hat{\psi}}(\tau) \int_0^\tau \dot{\hat{\psi}}^{-1}(s) \|H\|_{L^\infty(s,\infty)} \, ds$. y is the solution of

$$\dot{y} = \frac{\hat{n} - \hat{p}}{\dot{\hat{\psi}}} y + \|H\|_{L^\infty(s,\infty)}, \quad y(0) = 0.$$

An application of Lemma A.d.1 with $\mu = \dot{\hat{\psi}}^{-1}(0)$ gives

$$|y(\tau)| \leqslant \dot{\hat{\psi}}(0) \exp\left(-\dot{\hat{\psi}}^{-1}(0)\frac{\tau}{2}\right)\|H\|_{L^\infty(0,\infty)} + \dot{\hat{\psi}}(0)\|H\|_{L^\infty(\tau/2,\infty)}$$

Using this estimate in (A.d.10) implies

$$\|\hat{\psi}_p\|_{L^\infty(2i\tau_0,\infty)} \leqslant \text{const } \lambda^i \ln\sqrt{b} \left[\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{b}(\ln\sqrt{b})^{3/2} + \ln\frac{1}{\lambda}\left(\frac{1}{\sqrt{b-1}} + \delta^4(\ln\sqrt{b})^{3/2}\right)\right]$$

$$\leqslant \text{const } \lambda^i\left[\ln\frac{1}{\lambda}\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right]$$

$$\|\dot{\hat{\psi}}_p\|_{L^\infty(2i\tau_0,\infty)} \leqslant \text{const } \lambda^i\left[\sqrt{\frac{b+1}{b-1}}\left(\ln\frac{1}{\lambda}\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right) + \right.$$

$$\left. + \sqrt{\ln\sqrt{b}}\ \lambda^i\ln\frac{1}{\lambda}\left(\frac{1}{\sqrt{b-1}} + \delta^4(\ln\sqrt{b})^{3/2}\right)\right] \leqslant$$

$$\leqslant \text{const } \lambda^i\left(\frac{1}{b}\ln\frac{1}{\lambda} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right)$$

These estimates enable us to show the assertions of Lemma A.d.19:

$$\|\hat{\psi}_1\|_{L^\infty(2i\tau_0,\infty)} \leqslant \text{const}\left[\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ln\sqrt{b}}\right)\frac{1}{\sqrt{\ln\sqrt{b}}}\lambda^{2i}\dot{\hat{\psi}}(0) + \right.$$

$$\left. + \lambda^i\left(\ln\frac{1}{\lambda}\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right)\right],$$

$$\|\dot{\hat{\psi}}_1\|_{L^\infty(2i\tau_0,\infty)} \leqslant \text{const}\left[\left(\frac{1}{b} + \delta^4\right)\lambda^{2i}\frac{b}{b-1}\dot{\hat{\psi}}^2(0) + \lambda^i\left(\frac{1}{b}\ln\frac{1}{\lambda} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right)\right],$$

$$\|\hat{n}_1\|_{L^\infty(2i\tau_0,\infty)} \leq \text{const} \frac{b}{b-1} \lambda^i\left(\ln\frac{1}{\lambda}\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right),$$

$$\|\hat{p}_1\|_{L^\infty(2i\tau_0,\infty)} \leq \text{const} \frac{1}{b-1} \lambda^i\left(\ln\frac{1}{\lambda}\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{b}\ (\ln\sqrt{b})^{5/2}\right).$$

The last Lemma of this paragraph gives estimates on the smooth terms in the asymptotic expansion.

Lemma A.d.20: Let $0 < \bar{x} < 1$. Then

a) $\|\bar{p}'\|_{L^\infty(0,\bar{x})} \leq \text{const}\left(\frac{1}{b-1} + \delta^4\right),$

b) $\|\bar{p}'\|_{L^\infty(0,1)} \leq \text{const}\left(\frac{b}{(b-1)^2} + \delta^4\right),$

c) $\|\bar{p}''\|_{L^\infty(0,\bar{x})} \leq \text{const}\left(\frac{1}{b-1} + \delta^8\right),$

d) $\|\bar{p}''\|_{L^\infty(0,1)} \leq \text{const}\left(\frac{b^2}{(b-1)^4} + \delta^8\right),$

d) $\|\bar{p}'_1\|_{L^\infty(0,\bar{x})} \leq \text{const}\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ln\sqrt{b}}\right),$

f) $\|\bar{p}'_1\|_{L^\infty(0,1)} \leq \text{const}\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{\ln\sqrt{b}}\right)$

g) $\|\bar{p}_1\|_{L^\infty(0,1)} \leq \text{const}\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ln\sqrt{b}}\right),$

h) $\|\bar{p}''_1\|_{L^\infty(0,1)} \leq \text{const}\left(\frac{1}{(b-1)^{5/2}} + \delta^8\sqrt{\ln\sqrt{b}}\right).$

Proof: We will use the estimates

$$(A.d.11) \qquad \left|\frac{1}{2\bar{p}(x) + 1}\right| \leq \frac{b-1}{(b+1)\sqrt{1-x}},$$

$$(A.d.12) \qquad \left\|\frac{1}{2\bar{p}+1}\right\|_{L^\infty(0,\bar{x})} \leq \text{const}\frac{b-1}{b+1}, \qquad \left\|\frac{1}{2\bar{p}+1}\right\|_{L^\infty(0,1)} \leq \text{const},$$

which we obtain from (2.8) by elementary calculations. From (2.3) and (2.8) we obtain

$$\|\bar{p}'\|_{L^\infty(0,\bar{x})} = \left\|\frac{\bar{J}}{2(2\bar{p}+1)}\right\|_{L^\infty(0,\bar{x})} \leq \text{const}\left(\frac{b}{(b-1)^2} + \delta^4\right)\frac{b-1}{b+1} \leq \text{const}\left(\frac{1}{b-1} + \delta^4\right),$$

$$\|\bar{p}'\|_{L^\infty(0,1)} \leq \text{const}\left(\frac{b}{(b-1)^2} + \delta^4\right),$$

$$\|\bar{p}''\|_{L^\infty(0,\bar{x})} = \|\frac{\bar{J}\bar{p}'}{(2\bar{p}+1)^2}\|_{L^\infty(0,\bar{x})} < \text{const}\left(\frac{b}{(b-1)^2} + \delta^4\right)\left(\frac{b-1}{b+1}\right)^2\left(\frac{1}{b-1} + \delta^4\right) <$$

$$< \text{const}\left(\frac{1}{b-1} + \delta^8\right) ,$$

$$\|\bar{p}''\|_{L^\infty(0,1)} < \text{const}\left(\frac{b}{(b-1)^2} + \delta^4\right)\left(\frac{b}{(b-1)^2} + \delta^4\right) < \text{const}\left(\frac{b^2}{(b-1)^4} + \delta^8\right) .$$

The assertions on $\bar{p}_1(x) = -\bar{J}\dot{\psi}(0) \dfrac{(b-1)(1-x)}{(b+1)2(2\bar{p}(x)+1)}$ remain to be shown:

$$\|\bar{p}_1\|_{L^\infty(0,1)} < \text{const}|\bar{J}|\dot{\psi}(0) \frac{b-1}{b+1} \|\sqrt{1-x}\frac{b-1}{b+1}\|_{L^\infty(0,1)} <$$

$$< \text{const}\left(\frac{b}{(b-1)^2} + \delta^4\right)\sqrt{\ln\sqrt{b}}\left(\frac{b-1}{b+1}\right)^2 < \text{const}\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ln\sqrt{b}}\right) ,$$

$$|\bar{p}_1'(x)| = |\bar{J}\dot{\psi}(0) \frac{b-1}{2(b+1)(2\bar{p}(x)+1)}\left(1 - \frac{(1-x)\bar{J}}{(2\bar{p}+1)^2}\right)| <$$

$$< \text{const}|\bar{J}|\dot{\psi}(0) \frac{b-1}{b+1} |\frac{1}{2\bar{p}(x)+1}|$$

by (A.d.11). Thus,

$$\|\bar{p}_1'\|_{L^\infty(0,\bar{x})} < \text{const}\left(\frac{b}{(b-1)^2} + \delta^4\right)\sqrt{\ln\sqrt{b}}\left(\frac{b-1}{b+1}\right)^2 < \text{const}\left(\frac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ln\sqrt{b}}\right) ,$$

$$\|\bar{p}_1'\|_{L^\infty(0,1)} < \text{const}\left(\frac{1}{\sqrt{b-1}} + \delta^4\sqrt{\ln\sqrt{b}}\right) ,$$

$$\|\bar{p}_1''\|_{L^\infty(0,1)} = \|\bar{J}^2\dot{\psi}(0) \frac{b-1}{(b+1)(2\bar{p}+1)^3}\left(1 - \frac{3(1-x)\bar{J}}{2(2\bar{p}+1)^2}\right)\|_{L^\infty(0,1)} <$$

$$< \text{const}\left(\frac{b^2}{(b-1)^4} + \delta^8\right)\sqrt{\ln\sqrt{b}}\frac{b-1}{b+1} < \text{const}\left(\frac{1}{(b-1)^{5/2}} + \delta^8\sqrt{\ln\sqrt{b}}\right)$$

APPENDIX B: ESTIMATES OF THE RESIDUAL

Substitution of the formal asymptotic approximation

$$\bar{y} = (\bar{\psi} + \hat{\psi} + \lambda\bar{\psi}_1 + \lambda\hat{\psi}_1, \; \bar{n} + \hat{n} + \lambda\bar{n}_1 + \lambda\hat{n}_1, \; \bar{p} + \hat{p} + \lambda\bar{p}_1 + \lambda\hat{p}_1, \; \bar{J} + \lambda\bar{J}_1)$$

into problem (3.1a) gives the residual

$$
\begin{aligned}
F_{\lambda,\delta,b}(\bar{y}) = \Big(&\lambda^2\bar{\psi}'' + \lambda^3\bar{\psi}_1''', \; -\frac{1}{\lambda}(\bar{n} - \bar{n}(0) - x\bar{n}'(0))\dot{\hat{\psi}} + (\bar{n}(0) - \bar{n})\dot{\hat{\psi}}_1 + \\
&+ (\bar{n}_1(0) - \bar{n}_1)\dot{\hat{\psi}} + (\bar{\psi}'(0) - \bar{\psi}')\hat{n} - \lambda[\bar{\psi}'\hat{n}_1 + \bar{\psi}_1'\hat{n} + (\bar{n}_1 + \hat{n}_1)\dot{\hat{\psi}}_1] - \\
&- \lambda^2\bar{\psi}_1'(\bar{n}_1 + \hat{n}_1), \; \frac{1}{\lambda}(\bar{p} - \bar{p}(0) - x\bar{p}'(0))\dot{\hat{\psi}} + (\bar{p} - \bar{p}(0))\dot{\hat{\psi}}_1 + \\
&+ (\bar{p}_1 - \bar{p}_1(0))\dot{\hat{\psi}} + (\bar{\psi}' - \bar{\psi}'(0))\hat{p} + \lambda[\bar{\psi}'\hat{p}_1 + \bar{\psi}_1'\hat{p} + (\bar{p}_1 + \hat{p}_1)\dot{\hat{\psi}}_1] + \\
&+ \lambda^2\bar{\psi}_1'(\bar{p}_1 + \hat{p}_1), \, 0, \hat{\psi}(\lambda^{-1}) + \lambda\hat{\psi}_1(\lambda^{-1}), 0, \hat{n}(\lambda^{-1}) + \lambda\hat{n}_1(\lambda^{-1}), \hat{p}(\lambda^{-1}) + \lambda\hat{p}_1(\lambda^{-1}) \Big) \; .
\end{aligned}
$$

We apply Lemma A.d.20 and obtain

$$\| \lambda^2\bar{\psi}'' \|_{L^\infty(0,1)} \leq \text{const } \lambda^2\Big(\frac{b^2}{(b-1)^4} + \delta^8\Big) \; ,$$

$$\| \lambda^2\bar{\psi}_1''' \|_{L^\infty(0,1)} \leq \text{const } \lambda^2\Big(\frac{1}{(b-1)^{5/2}} + \delta^8\sqrt{\ell n\sqrt{b}}\Big) \; .$$

We now apply Taylor's theorem and the Lemmas A.d.6 and A.d.20.

$$\| \frac{1}{\lambda}(\bar{n} - \bar{n}(0) - x\bar{n}'(0))\dot{\hat{\psi}} \|_{L^1(0,1)} \leq \frac{1}{\lambda}\| \bar{p} - \bar{p}(0) - x\bar{p}'(0) \|_{L^\infty(0,4\lambda\tau_0)} \| \lambda\dot{\hat{\psi}} \|_{L^1(0,\infty)} +$$

$$+ \frac{1}{\lambda}\| \bar{p} - \bar{p}(0) - x\bar{p}'(0) \|_{L^\infty(0,1)} \cdot \| \dot{\hat{\psi}} \|_{L^\infty(4\tau_0,\infty)} \leq$$

$$< const\left(\lambda^2 \tau_0^2 \|\bar{p}''\|_{L^\infty(0,4\lambda\tau_0)} |\dot{\hat{\psi}}(0)| + \|\bar{p}\|_{L^\infty(0,1)} \cdot \lambda^3 \dot{\hat{\psi}}(0)\right) <$$

$$< const \; \lambda^2 \left(\ln \tfrac{1}{\lambda}\right)^2 \left(\frac{b-1}{b^2} + \delta^8 (\ln\sqrt{b})^2\right) .$$

Similarly we obtain

$$\|(\bar{n}(0) - \bar{n})\dot{\hat{\psi}}_1\|_{L^1(0,1)} < \|\bar{p}(0) - \bar{p}\|_{L^\infty(0,6\lambda\tau_0)} \lambda \|\dot{\hat{\psi}}_1\|_{L^1(0,\infty)} +$$

$$+ \|\bar{p}(0) - \bar{p}\|_{L^\infty(0,1)} \|\dot{\hat{\psi}}_1\|_{L^\infty(6\tau_0,\infty)} < 6\lambda^2\tau_0 \|\bar{p}'\|_{L^\infty(0,6\lambda\tau_0)} |\hat{\psi}_1(0)| +$$

$$+ \|\bar{p}'\|_{L^\infty(0,1)} const \; \lambda^3 \left(\tfrac{1}{b} \ln \tfrac{1}{\lambda} + \delta^4\sqrt{b} \; (\ln\sqrt{b})^{5/2}\right)$$

$$< const \; \lambda^2 \ln \tfrac{1}{\lambda} \sqrt{\ln\sqrt{b}} \left(\tfrac{1}{b-1} + \delta^4\right)\left(\tfrac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ln\sqrt{b}}\right) <$$

$$< const \; \lambda^2 \ln \tfrac{1}{\lambda} \left(\tfrac{1}{b} + \delta^8 \ln\sqrt{b}\right) ,$$

$$\|(\bar{n}_1(0) - \bar{n}_1)\dot{\hat{\psi}}\|_{L^1(0,1)} < 3\lambda\tau_0 \|\bar{p}_1'\|_{L^\infty(0,3\lambda\tau_0)} \lambda \|\dot{\hat{\psi}}\|_{L^1(0,\infty)} + \|\bar{p}_1'\|_{L^\infty(0,1)} \|\dot{\hat{\psi}}\|_{L^\infty(3\tau_0,\infty)}$$

$$< const\left[\lambda^2 \ln \tfrac{1}{\lambda} \sqrt{\ln\sqrt{b}} \left(\tfrac{\sqrt{b-1}}{b} + \delta^4\sqrt{\ln\sqrt{b}}\right)\ln\sqrt{b} + \left(\tfrac{1}{\sqrt{b-1}} + \delta^4\sqrt{\ln\sqrt{b}}\right)\lambda^3\sqrt{\ln\sqrt{b}}\right]$$

$$< const \; \lambda^2 \ln \tfrac{1}{\lambda} \delta^4 (\ln\sqrt{b})^2 ,$$

$$\|(\bar{\psi}'(0) - \bar{\psi}')\hat{n}\|_{L^1(0,1)} < 3\lambda\tau_0 \|\bar{\psi}''\|_{L^\infty(0,3\lambda\tau_0)} \lambda\|\hat{n}\|_{L^1(0,\infty)} + \|\bar{\psi}''\|_{L^\infty(0,1)} \|\hat{n}\|_{L^\infty(3\tau_0,\infty)}$$

$$< const\left[\lambda^2 \ln \tfrac{1}{\lambda} \sqrt{\ln\sqrt{b}} \left(\tfrac{1}{b-1} + \delta^8\right)\sqrt{\ln\sqrt{b}} + \left(\tfrac{b^2}{(b-1)^4} + \delta^8\right)\lambda^3 \tfrac{b}{b-1} \ln\sqrt{b}\right]$$

$$< const \; \lambda^2 \ln \tfrac{1}{\lambda} \left(\tfrac{1}{b} + \delta^8 \ln\sqrt{b}\right)$$

$$\|\lambda\bar{\psi}'\hat{n}_1\|_{L^1(0,1)} \leq \lambda\|\bar{\psi}'\|_{L^\infty(0,4\lambda\tau_0)}\,4\lambda\tau_0\|\hat{n}_1\|_{L^\infty(0,\infty)} + \lambda\|\bar{\psi}'\|_{L^\infty(0,1)}\|\hat{n}_1\|_{L^\infty(4\tau_0,\infty)} \leq$$

$$\leq const[\lambda^2(\frac{1}{b-1}+\delta^4)\ln\frac{1}{\lambda}\sqrt{\ln\sqrt{b}}\,(\frac{1}{\sqrt{b-1}}+\delta^4\sqrt{b}\,(\ln\sqrt{b})^{5/2}) +$$

$$+ \lambda^3(\frac{b}{(b-1)^2}+\delta^4)(\frac{1}{\sqrt{b-1}}\ln\frac{1}{\lambda}+\delta^4\sqrt{b}\,(\ln\sqrt{b})^{5/2})] \leq$$

$$\leq const\,\lambda^2\ln\frac{1}{\lambda}\,(\frac{1}{b-1}+\delta^8\sqrt{b}\,(\ln\sqrt{b})^3)\;,$$

$$\|\lambda\bar{\psi}_1'\hat{n}\|_{L^1(0,1)} \leq \lambda^2\|\bar{\psi}_1'\|_{L^\infty(0,1)}\|\hat{n}\|_{L^1(0,\infty)} \leq const\,\lambda^2(\frac{1}{\sqrt{b-1}}+\delta^4\sqrt{\ln\sqrt{b}})\,\sqrt{\ln\sqrt{b}} \leq$$

$$\leq const\,\lambda^2(\frac{1}{b}+\delta^4\ln\sqrt{b})\;,$$

$$\|\lambda(\bar{n}_1+\hat{n}_1)\dot{\bar{\psi}}_1\|_{L^1(0,1)} \leq \lambda^2\|\bar{p}_1+\hat{n}_1\|_{L^\infty(0,4\lambda\tau_0)}\,4\tau_0\|\dot{\bar{\psi}}_1\|_{L^\infty(0,\infty)} +$$

$$+ \lambda\|\bar{n}_1+\hat{n}_1\|_{L^\infty(4\lambda\tau_0,1)}\|\dot{\bar{\psi}}_1\|_{L^\infty(4\tau_0,\infty)} \leq const[\lambda^2(\frac{\sqrt{b-1}}{b}+\delta^4\sqrt{\ln\sqrt{b}}+\frac{1}{\sqrt{b-1}}+$$

$$+ \delta^4\sqrt{b}\,(\ln\sqrt{b})^{5/2})\ln\frac{1}{\lambda}\sqrt{\ln\sqrt{b}}\,(\frac{1}{b}+\delta^4\sqrt{b}\,(\ln\sqrt{b})^{5/2})$$

$$+ \lambda^3(\frac{\sqrt{b-1}}{b}+\delta^4\sqrt{\ln\sqrt{b}}+\lambda^2(\frac{1}{\sqrt{b-1}}\ln\frac{1}{\lambda}+\delta^4\sqrt{b}\,(\ln\sqrt{b})^{5/2}))(\frac{1}{b}\ln\frac{1}{\lambda}+\delta^4\sqrt{b}\,(\ln\sqrt{b})^{5/2})]$$

$$\leq const\,\lambda^2\ln\frac{1}{\lambda}\,(\frac{1}{b}+\delta^8 b(\ln\sqrt{b})^{11/2})\;,$$

$$\|\lambda^2\bar{\psi}_1'(\bar{n}_1+\hat{n}_1)\|_{L^1(0,1)} \leq \lambda^2\|\bar{\psi}_1'\|_{L^\infty(0,1)}(\|\bar{p}_1\|_{L^\infty(0,1)}+\|\hat{n}_1\|_{L^\infty(0,\infty)}) \leq$$

$$\leq const\,\lambda^2(\frac{1}{\sqrt{b-1}}+\delta^4\sqrt{\ln\sqrt{b}})(\frac{\sqrt{b-1}}{b}+\delta^4\sqrt{\ln\sqrt{b}}+\frac{1}{\sqrt{b-1}}+\delta^4\sqrt{b}\,(\ln\sqrt{b})^{5/2}) \leq$$

$$\leq const\,\lambda^2(\frac{1}{b-1}+\delta^8\sqrt{b}\,(\ln\sqrt{b})^3)\;.$$

The remaining terms are treated analogously, and

$$(B.1) \qquad |F_{\lambda,\delta,b}(\bar{y})|_{2,\lambda} \leqslant \text{const } \lambda^2 \Big[\frac{1}{(b-1)^4} + \ln \frac{1}{\lambda} \, \delta^8 b (\ln \sqrt{b})^{11/2} \Big]$$

follows.

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In this paper the basic semiconductor device equations modelling a symmetric one-dimensional voltage-controlled diode are formulated as a singularly perturbed two point boundary value problem. The perturbation parameter is the normed Debye-length of the device. We derive the zeroth and first order terms of the matched asymptotic expansion of the solutions, which are the sums of uniformly smooth outer terms (reduced solutions) and the exponentially varying inner terms (layer solutions). The main result of the

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ABSTRACT (cont.)

paper is that, if the perturbation parameter is sufficiently small then there
exists a solution of the semiconductor device problem which is approximated
uniformly by the zeroth order term of the expansion, even for large applied
voltages. This result shows the validity of the asymptotic expansions of the
solutions of the semiconductor device problem in physically relevant high-
injection conditions.

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